Sine and Cosine Laws from Vector and Scalar Products

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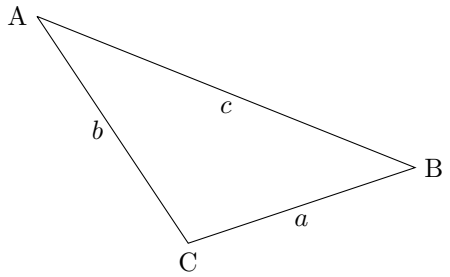
Abstract

The familiar sine and cosine laws can be obtained using the vector and scalar products of two vectors.

1 Introduction

Consider triangle ABC as shown:



Considering this from a vector perspective we have:

$$\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{AC}} - \overrightarrow{\mathrm{BC}}$$

2 Cosine Law

We start with the vector equation in the introduction and take the scalar product of each side of the equation with itself. We use the rule that the scalar product of any vector with itself is the square of the vector's magnitude. We also assume the distributive property of the scalar product.

$$\begin{aligned}
\overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{AB}} &= (\overrightarrow{\mathrm{AC}} - \overrightarrow{\mathrm{BC}}) \cdot (\overrightarrow{\mathrm{AC}} - \overrightarrow{\mathrm{BC}}) \\
|\overrightarrow{\mathrm{AB}}|^2 &= \overrightarrow{\mathrm{AC}} \cdot \overrightarrow{\mathrm{AC}} - 2\overrightarrow{\mathrm{AC}} \cdot \overrightarrow{\mathrm{BC}} + \overrightarrow{\mathrm{BC}} \cdot \overrightarrow{\mathrm{BC}} \\
|\overrightarrow{\mathrm{AB}}|^2 &= |\overrightarrow{\mathrm{BC}}|^2 + |\overrightarrow{\mathrm{AC}}|^2 - 2\overrightarrow{\mathrm{AC}} \cdot \overrightarrow{\mathrm{BC}} \\
c^2 &= a^2 + b^2 - 2ab\cos C
\end{aligned}$$

3 Sine Law

We start with the vector equation in the introduction and take the vector product of each side of the equation with $\overrightarrow{\mathrm{AB}}$. We use the rule that the vector product of any vector with itself is the zero vector. We also assume the distributive property of the vector product. (Strictly speaking, this presupposes that we are working with vectors in three dimensions since the vector product is not defined for vectors in the plane. A conclusion for a triangle in space, however, will also hold for a triangle in the plane.)

$$\begin{aligned}
\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AB}} &= (\overrightarrow{\mathrm{AC}} - \overrightarrow{\mathrm{BC}}) \times \overrightarrow{\mathrm{AB}} \\
\vec{0} &= \overrightarrow{\mathrm{AC}} \times \overrightarrow{\mathrm{AB}} - \overrightarrow{\mathrm{BC}} \times \overrightarrow{\mathrm{AB}} \\
\overrightarrow{\mathrm{AC}} \times \overrightarrow{\mathrm{AB}} &= \overrightarrow{\mathrm{BC}} \times \overrightarrow{\mathrm{AB}} \\
|\overrightarrow{\mathrm{AC}} \times \overrightarrow{\mathrm{AB}}| &= |\overrightarrow{\mathrm{BC}} \times \overrightarrow{\mathrm{AB}}| \\
|\overrightarrow{\mathrm{AC}}|\,|\overrightarrow{\mathrm{AB}}| \sin \angle \mathrm{BAC} &= |\overrightarrow{\mathrm{BC}}|\,|\overrightarrow{\mathrm{AB}}| \sin \angle \mathrm{ABC} \\
|\overrightarrow{\mathrm{AC}}| \sin \angle \mathrm{BAC} &= |\overrightarrow{\mathrm{BC}}| \sin \angle \mathrm{ABC} \\
b \sin A &= a \sin B \\
\frac{b}{\sin B} &= \frac{a}{\sin A}
\end{aligned}$$