We have been considering solving simultaneous equations and ended up with the following system as an example having infinite solutions.

$$\begin{aligned} x + y - 2z &= -4 \\ 2x + y - z &= -1 \\ x - 2y + 7z &= 17 \end{aligned}$$

Once we have determined that the three planes all intersect in a single line, the obvious next step is to determine an equation for that line. We can provide this in vector terms. (Remember that a line in space cannot be represented by a single Cartesian equation. It requires at least two equations or a set of parametric equations.)

First represent the equations as vector equations of the three planes they represent:

$$\begin{aligned} \mathbf{r}_1 \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} &= -4 \\ \mathbf{r}_2 \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} &= -1 \\ \mathbf{r}_3 \cdot \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} &= 17 \end{aligned}$$

These are in normal form, and every line in each plane must be perpendicular to its normal, so the intersection of two planes must be perpendicular to their two normals. Hence the direction of the line of intersection can be determined using cross product. Taking the first two equations,

$$\begin{aligned} \mathbf{d} &= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} \end{aligned}$$

Taking the first and third,

$$\begin{aligned} \mathbf{d} &= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -9 \\ -3 \end{bmatrix} \end{aligned}$$

And finally the second and third,

$$\begin{aligned}\mathbf{d} &= \begin{bmatrix}2\\1\\-1\end{bmatrix} \times \begin{bmatrix}1\\-2\\7\end{bmatrix}\\ &= \begin{bmatrix}5\\-15\\-5\end{bmatrix}\end{aligned}$$

We can see that all these cross products are scalar multiples of each other, indicating that the intersection lines between pairs of planes are all parallel. If they were three distinct lines (as in the previous example that had the same left hand sides but a different right hand side for equation 3), then the system has no solution: pairs of planes meet in parallel but distinct lines. But if (as here) the lines coincide then the three planes meet in a single line and every point on the line is a solution to the system of equations.

We can choose any of these cross products for the direction of our line: we will use the first.

To compete the equation of the line, all that remains is to find any point on the line. Row reduction got the augmented matrix to

$$\begin{bmatrix}1 & 1 & -2 & -4\\0 & -1 & 3 & 7\\0 & 0 & 0 & 0\end{bmatrix}$$

The equation represented by the third row will be true for any value of $z$ so we can choose any value convenient. Let's use $z = 0$. Substituting into the equation represented by the second row gives us

$$\begin{aligned}-y + 3(0) &= 7\\ y &= -7\end{aligned}$$

The first equation, then, gives

$$\begin{aligned}x + (-7) - 2(0) &= -4\\ x &= 3\end{aligned}$$

Thus we have the point $(3, -7, 0)$ satisfying all three equations simultaneously: it is a point on the line. The equation of the line, then is

$$\mathbf{r} = \begin{bmatrix}3\\-7\\0\end{bmatrix} + \lambda \begin{bmatrix}1\\-3\\-1\end{bmatrix}$$

Note that choosing a different value for $z$ gives us a different initial point, but the same line. Try it with $z = 1$:

$$\begin{aligned}-y + 3(1) &= 7\\ y &= -4\\ x + (-4) - 2(1) &= -4\\ x &= 2\end{aligned}$$

Thus we have the point $(2, -4, 1)$, and it can be seen that this is a point on the line

$$\mathbf{r} = \begin{bmatrix} 3 \\ -7 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$$

where $\lambda = -1$.