Topics in Secondary Mathematics

Matrices

Glen Prideaux

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MATRICES
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ISBN 978-1-326-91164-5
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2017

Published by Glen Prideaux, using Lulu.com

Typeset in Computer Modern using LaTeX. LaTeXis without question the best option available for generating math-heavy documents. Students of mathematics, learn LaTeX!

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For my students

Contents

Preface

I have often found myself saying to teacher colleagues that what I *really* want from a text book is a set of well designed, graded practice problems for students to work through. I don't need the book to contain explanations and examples; I'll give my students what explanations and examples they need, and if they want more there are numerous places they can go on the Internet to get more. *Topics in Secondary Mathematics* sets out to be such a resource. I intend to include a large number of questions of graded difficulty and complexity with answers to odd numbered questions (so students can get immediate feedback while also allowing teachers to validate students' work) and some fully worked solutions.

The contents of this book are influenced by the Australian curriculum, but no attempt has been made to follow any specific curriculum boundaries. This is a deliberate attempt to help teachers avoid the temptation of teaching to a text book rather than the official curriculum.

ACKNOWLEDGEMENTS

Thanks go firstly to my wife Carol for her unwavering encouragement and support. Thanks also to colleagues who have provided encouragement. Thanks especially to my students who have used this resource and have helped to identify and correct errors.

1 Matrix Basics

1.1 Representing Information in Matrices

1. The N-P-K values for some fertilizer types are:

- Bird guano: N:16, P:12, K:3;
- Poultry manure: N:3, P:2, K:2;
- Bone meal: N:4, P:12, K:0;
- Ammonium Sulfate: N:21, P:0, K:0.

Represent this information in a matrix, with rows representing the elements N, P and K, and columns representing the different types of fertilizer.

2. Nutritional information for some breakfast ingredients includes the following per 100g:

- Cereal W has 12.4g of protein, 67g of carbohydrates and 1.4g of fat;
- Whole milk has 3.2g fat, 3.2g protein and 4.8g carbohydrates;
- Banana has 23g carbohydrates, 0.3g fat and 1.1g protein;
- Croissant has 21g fat, 46g carbohydrates and 8g protein.

Represent this information in a matrix, with rows representing different ingredients and columns representing the nutrients.

3. A researcher observes interactions in a small colony of rodents and records the number of interactions as follows:

- Animal A interacts with animal B five times and with animal C 12 times.
- Animal B also interacts with animal C 15 times and with animal D three times.
- Animal C also interacts with animal D twice.

Represent this information in a matrix.

4. A teacher records how often the students in her class work together during a term.

- Aiden works with Beth 18 times, with Connor six times, with Drishti four times and with Emily twice.
- Beth works with Aiden 18 times, with Drishti six times, and with Emily six times.
- Conor works with Aiden six times, with Drishti twice and with Emily 22 times.
- Drishti works with Aiden four times, with Beth six times, with Connor twice.

Emily's record has been misplaced, but there should be enough information present anyway. Represent the information in a matrix.

5. In a Christmas gift exchange, Jordan gives a gift each to Liam and Natalie; Kelsey gives a gift each to Liam, Matt, Natalie and Owen; Liam gives a gift each to Kelsey, Natalie and Owen; Matt gives a gift each to Kelsey and Natalie; Natalie gives a gift to Kelsey; and Owen gives one gift each to Jordan, Liam and Matt, and two gifts to Natalie. Represent this information in a matrix.

6. In a popular (and bloody) fantasy epic,

- Gregor killed Oberyn.
- Gregor killed Beric twice.
- Sandor killed Beric.
- Oberyn killed Gregor.
- Petyr killed Jon.
- Lyssa killed Jon.
- Petyr killed Lyssa.

Represent this information in a matrix.

7. A particular dairy farmer's herd is made up of cows of different ages: 25 calves under a year old, 21 year-old heifers, 19 two-year-old cows, 28 three-year-old cows, 15 four-year-old cows and 5 five-year-old cows. Represent this information in a row matrix.

8. A particular secondary school has 125 year 7s, 142 year 8s, 139 year 9s, 112 year 10s, 121 year 11s and 108 year 12s. Represent this information in a row matrix.

9. In the book 101 Dalmations, Pongo and Missus's litter had eight male puppies and seven female. Represent this information in a column matrix.

10. Harrison's stud farm has five chestnut horses, two blacks, three greys and two palamino. Represent this information in a column matrix.

11. The matrix below shows the number of students in different year groups for each of three houses in a secondary school, represented by the letters C, M and P.

$$\begin{array}{c c} Yr & \begin{array}{ccc} C & M & P \end{array} \\ \begin{array}{c} 12 \\ 11 \\ 10 \\ 9 \\ 8 \\ 7 \end{array} & \begin{bmatrix} 43 & 39 & 40 \\ 36 & 39 & 37 \\ 46 & 44 & 39 \\ 40 & 39 & 44 \\ 32 & 32 & 31 \\ 34 & 34 & 36 \end{bmatrix} \end{array}$$

a) How many year nine students are in house M?
b) How many year 12 students are there altogether?
c) What fraction of house C students are in year 11?
d) What fraction of the total are in house P?

12. The matrix shows numbers of students and staff participating in Arts Day activities at a secondary school. The activities all run at the same time.

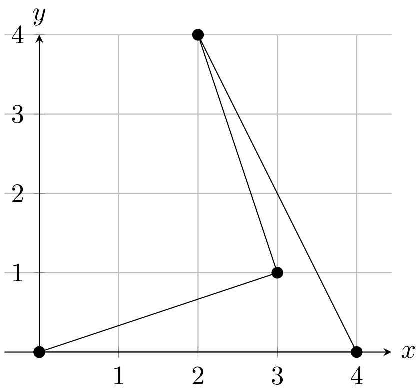
$$\begin{array}{c c} \text{Event} & \begin{array}{cc} \text{Students} & \text{Staff} \end{array} \\ \begin{array}{c} \text{Drama Performance} \\ \text{Mask Making} \\ \text{Instrument Making} \\ \text{African Drumming} \\ \text{Samba Drumming} \\ \text{Hip-Hop} \\ \text{Circus Skills} \end{array} & \begin{bmatrix} 400 & 19 \\ 20 & 6 \\ 20 & 4 \\ 15 & 5 \\ 15 & 2 \\ 30 & 4 \\ 20 & 5 \end{bmatrix} \end{array}$$

a) How many staff are attending the Drama Performance?

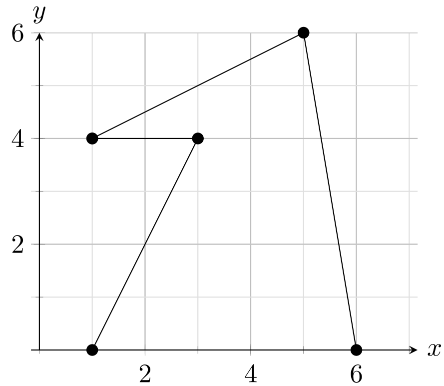
b) How many people does the venue for Mask Making need to be able to accommodate?
c) How many staff are there altogether?
d) What fraction of students selected Circus Skills as their activity?

13. Make a matrix of the coordinates of the vertices of the following diagram, with the first row being x-coordinates and the second row y-coordinates.

Multiple correct answers are possible. Your answer will depend on which point you start at, and which direction you go.



14. Make a matrix of the coordinates of the vertices of the following diagram, with the first row being x-coordinates and the second row y-coordinates.



15. The following matrix represents a simple diagram defined by points on the Cartesian plane, with the first row being x-coordinates and the second row y-coordinates. Plot the points and join them in order to make the diagram.

$$\begin{bmatrix} 1 & 1 & 3.5 & 4 & 2 & 4 & 4 & 5 & 3 & 4 & 4 & 3 & 0.5 \\ 0 & 2 & 2 & 3 & 4 & 4 & 5 & 5 & 8 & 8.5 & 11 & 12 & 12 \end{bmatrix}$$

16. The following matrix represents a simple diagram defined by points on the Cartesian plane, with the first row being x-coordinates and the second row y-coordinates. Plot the points and join them in order to make the diagram.

$$\begin{bmatrix} 1 & 1 & 0 & 5 & 7 & 7 & 8 & 8 & 10 & 9 & 9 & 1 \\ 1 & 4 & 4 & 6 & 5.2 & 6 & 6 & 4.8 & 4 & 4 & 1 & 1 \end{bmatrix}$$

1.2 Matrix Size

Give the size of the following matrices:

17. $\begin{bmatrix} 1 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$

19. $\begin{bmatrix} -4 & 5 \\ 1 & -9 \end{bmatrix}$

20. $\begin{bmatrix} -3 & -9 & -3 \\ -4 & -5 & -6 \\ -8 & -1 & 6 \\ 4 & 5 & -9 \end{bmatrix}$

21. $\begin{bmatrix} -6 \\ 8 \\ 6 \\ 6 \\ 9 \end{bmatrix}$

22. $\begin{bmatrix} -1 & -6 \\ -8 & -2 \\ -1 & 1 \\ 6 & -9 \end{bmatrix}$

23. $\begin{bmatrix} -8 & 8 & 8 & -7 & 8 & 5 & 4 & -4 \\ -3 & 0 & -9 & 4 & 5 & 6 & -2 & -9 \end{bmatrix}$

24. $\begin{bmatrix} 8 & 4 & 9 & 0 & -7 \\ -8 & -2 & -9 & -5 & -5 \\ 6 & 8 & 1 & -8 & -3 \\ 1 & 6 & -8 & -4 & -7 \\ -7 & -7 & -1 & 3 & -5 \\ 6 & 10 & 10 & -1 & -4 \end{bmatrix}$

1.3 Addressing Matrix Elements

25. Given $A = \begin{bmatrix} 1 & 0 & 9 & -5 \\ -7 & -3 & -8 & 2 \\ 5 & -9 & 4 & 8 \end{bmatrix}$, give the value (where possible) of the elements listed below.

a) $A_{3,3}$

b) $A_{4,2}$

c) $A_{3,2}$

d) $A_{2,4}$

e) $A_{2,0}$
f) $A_{2,1}$
g) $A_{1,2}$
h) $A_{1,3}$

26. Given $B = \begin{bmatrix} -4.1 & 7.8 & -2.3 \\ 2.4 & -2.4 & -2.0 \\ -4.7 & 0.1 & 4.7 \\ 3.0 & -3.2 & 1.7 \end{bmatrix}$, give the value (where possible) of the elements listed below.
a) $B_{4,3}$
b) $B_{3,3}$
c) $B_{4,4}$
d) $B_{3,5}$
e) $B_{1,3}$
f) $B_{1,1}$
g) $B_{3,2}$
h) $B_{2,2}$

27. Write a 3×3 matrix A where $A_{m,n} = 10n + m$

28. Write a 4×2 matrix B where $B_{m,n} = 10(m-1) + n$

29. Write a 5×3 matrix C where $C_{a,b} = ab$

30. Write a 2×6 matrix D where $D_{a,b} = \frac{2b}{a}$

31. Write a 4×4 matrix E where $E_{c,d} = c^2 + d$

32. Write a 3×5 matrix F where $F_{c,d} = c^2 - d^2$

1.4 Special Matrix Types

Classify each of the following matrices as square, column, row, identity, diagonal, zero or triangular. More than one (or none) may apply.

33. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

34. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

36. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

37. $\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$

38. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

39. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

40. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

41. $\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

42. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

43. $\begin{bmatrix} 1 \end{bmatrix}$

44. $\begin{bmatrix} -5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$

1.5 Matrix Row Operations

45. Given $C = \begin{bmatrix} 1 & 0 & 5 & 1 \\ 2 & -6 & -2 & 0 \\ -1 & -2 & 3 & 4 \end{bmatrix}$, give the result of the following sequential row operations:

a) Add row 1 to row 3.

b) Then add $-2\times$row 1 to row 2.

c) Then multiply row 3 by 3 and add $-1\times$row 2.

46. Given $D = \begin{bmatrix} 0 & 1 & 5 & 3 \\ 3 & 3 & 0 & 5 \\ 4 & 0 & 4 & -9 \end{bmatrix}$, give the result of the following sequential row operations:

a) Swap rows 1 and 2.

b) Then multiply row 3 by 3 and add $-4\times$row 1.

c) Then add $12\times$row 2 to row 3.

47. Given $E = \begin{bmatrix} 2 & -8 & 3 & 1 & -1 \\ 4 & -16 & 6 & 6 & -6 \\ 0 & 0 & 1 & 5 & 10 \\ 8 & -33 & 13 & 16 & 22 \end{bmatrix}$, give the result of the following sequential row operations:

a) Add $-2\times$row 1 to row 2.

b) Then add $-4\times$row 1 to row 4.

c) Then swap row 2 and row 4.

48. Given $F = \begin{bmatrix} 4 & -5 & -8 & -4 & -16 \\ 4 & -4 & -8 & 4 & -24 \\ 0 & 6 & 0 & 9 & -9 \\ 8 & -12 & 0 & -10 & 18 \end{bmatrix}$, give the result of the following sequential row operations:

a) Add $-1\times$row 1 to row 2.

b) Then add $-2\times$row 1 to row 4.

c) Then add $-6\times$row 2 to row 3.

d) Then add $2\times$row 2 to row 4.

e) Then swap row 3 and row 4.

49. Given $G = \begin{bmatrix} 1 & -5 & -7 & 1 & 12 \\ -5 & 10 & 7 & 2 & 26 \\ -2 & -2 & -1 & -7 & -59 \\ -8 & 5 & -6 & 2 & 47 \end{bmatrix}$, give the result of the following sequential row operations:

a) Add $5\times$row 1 to row 2.

b) Then add $2\times$row 1 to row 3.

c) Then add $8\times$row 1 to row 4.

d) Then multiply row 3 by 5 and add $-4\times$row 2.

e) Then multiply row 4 by 3 and add $-7\times$row 2.

f) Then multiply row 4 by 37 and add $-10\times$row 3.

50. Given $H = \begin{bmatrix} 3 & 8 & -7 & 10 & 7 \\ 0 & 1 & -1 & 4 & 10 \\ 6 & -1 & 1 & -5 & 16 \\ -8 & 5 & 4 & 5 & 8 \end{bmatrix}$, give the result of the following sequential row operations:

a) Add $-2\times$row 1 to row 3.
b) Then multiply row 4 by 3 and add $8\times$row 1 to row 4.
c) Then add $17\times$row 2 to row 3.
d) Then add $-79\times$row 2 to row 4.
e) Then multiply row 4 by 2 and add $35\times$row 3.

51. Use row operations to reduce $\begin{bmatrix} 1 & -4 & 1 \\ 4 & 10 & 0 \end{bmatrix}$ to row echelon form.

These questions have multiple correct answers.

52. Use row operations to reduce $\begin{bmatrix} -3 & -5 & -6 \\ -9 & -6 & -2 \end{bmatrix}$ to row echelon form.

53. Use row operations to reduce $\begin{bmatrix} 0 & 10 & 10 \\ 7 & 3 & -6 \end{bmatrix}$ to row echelon form.

54. Use row operations to reduce $\begin{bmatrix} 0 & 2 & 7 \\ 7 & -8 & 6 \end{bmatrix}$ to row echelon form.

55. Use row operations to reduce $\begin{bmatrix} 0 & 3 & -5 & -5 \\ 3 & 9 & 7 & 0 \\ -6 & -9 & 10 & 1 \end{bmatrix}$ to row echelon form.

56. Use row operations to reduce $\begin{bmatrix} 0 & -5 & 2 & -8 \\ 1 & -9 & 10 & 9 \\ 0 & 5 & 2 & 2 \end{bmatrix}$ to row echelon form.

57. Reduce $\begin{bmatrix} 1 & 1 & -6 & -2 \\ -6 & 4 & 7 & -6 \\ 9 & -1 & -6 & 10 \end{bmatrix}$ to row echelon form.

58. Reduce $\begin{bmatrix} 1 & 4 & 1 & 5 \\ -4 & -4 & -4 & -8 \\ 6 & 0 & -6 & 4 \end{bmatrix}$ to row echelon form.

59. Reduce $\begin{bmatrix} -8 & -7 & 4 & 10 \\ 8 & 1 & 3 & 7 \\ -8 & 3 & 9 & -8 \end{bmatrix}$ to row echelon form.

60. Reduce $\begin{bmatrix} -6 & -7 & 4 & -2 \\ -7 & 2 & -5 & 3 \\ 4 & -7 & 8 & 8 \end{bmatrix}$ to row echelon form.

61. Reduce $\begin{bmatrix} 1 & 2 & 1 & 2 & 9 \\ 2 & 4 & 2 & 2 & 1 \\ -8 & -6 & -3 & -7 & 7 \\ -4 & -5 & -9 & -1 & 8 \end{bmatrix}$ to row echelon form.

62. Reduce $\begin{bmatrix} 5 & -8 & 10 & 1 & -7 \\ 1 & -7 & 3 & -8 & -9 \\ -5 & 6 & 3 & 6 & -3 \\ 5 & 10 & 4 & 8 & 6 \end{bmatrix}$ to row echelon form.

63. Reduce $\begin{bmatrix} 2 & -2 & 1 & -8 & 1 & 0 \\ -2 & -3 & 3 & 8 & 9 & -7 \\ -6 & 6 & 1 & 16 & 4 & 6 \\ -6 & 1 & 9 & 1 & 19 & 0 \\ -4 & 4 & 6 & 0 & 4 & 10 \end{bmatrix}$ to R.E.F.

64. Reduce $\begin{bmatrix} -4 & -6 & 4 & 2 & 7 & 4 \\ 4 & 6 & -4 & -2 & -3 & 1 \\ -4 & -8 & 0 & -3 & -1 & 6 \\ 8 & 12 & -8 & 2 & -11 & -7 \\ 4 & 6 & 0 & 5 & 0 & 8 \end{bmatrix}$ to R.E.F.

65. Write $\begin{bmatrix} 4.10 & -1.42 & -6.01 & 0.89 \\ 5.52 & 6.57 & 4.37 & 9.88 \\ 0 & -8.12 & 4.73 & -8.69 \end{bmatrix}$ in row echelon form where the leading non-zero element of each row is 1, all rounded to two decimal places.

66. Write $\begin{bmatrix} -6.26 & 7.46 & -4.21 & 9.24 \\ -3.11 & -6.35 & -8.89 & -9.25 \\ -6.72 & -7.77 & 1.44 & 4.47 \end{bmatrix}$ in row echelon form where the leading non-zero element of each row is 1, all rounded to two decimal places.

67. Write $\begin{bmatrix} -8.9 & 6.9 & 3.9 & -6.8 & 31.5 \\ -9.1 & 3.3 & -6.4 & -7.8 & 45.2 \\ 1.8 & 2.0 & 2.2 & 7.1 & 85.2 \\ -2.7 & 6.2 & -0.4 & -5.5 & -48.6 \end{bmatrix}$ in row echelon form where the leading non-zero element of each row is 1, all rounded to two decimal places.

68. Write $\begin{bmatrix} -17.3 & 24.6 & 1.9 & 18.2 & -51.7 \\ -15.8 & 23.3 & 9.8 & 8.6 & -44.1 \\ -0.1 & 21.5 & -0.2 & 2.7 & -52.4 \\ -4.9 & 22.5 & -1.2 & 12.0 & 19.1 \end{bmatrix}$ in row echelon form where the leading non-zero element of each row is 1, all rounded to two decimal places.

1.6 Matrix Addition and Subtraction

69. Given the matrices listed below, decide whether each addition is possible.

$$A = \begin{bmatrix} a & b & c \end{bmatrix} \quad B = \begin{bmatrix} d \\ e \\ f \end{bmatrix} \quad C = \begin{bmatrix} g & h & i \\ j & k & l \end{bmatrix}$$

$$D = \begin{bmatrix} m & n \\ o & p \\ q & r \end{bmatrix} \quad E = \begin{bmatrix} s & t & u \\ v & w & x \\ y & z & \alpha \end{bmatrix}$$

Which sums are possible? $A+A$, $A+B$, $A+C$, $A+D$, $A+E$, $B+A$, $B+B$, $B+C$, $B+D$, $B+E$ $C+A$, $C+B$, $C+C$, $C+D$, $C+E$ $D+A$, $D+B$, $D+C$, $D+D$, $D+E$ $E+A$, $E+B$, $E+C$, $E+D$, or $E+E$?

70. Given the matrices listed below, decide whether each subtraction is possible.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which differences are possible? $A-A$, $A-B$, $A-C$, $A-D$, $A-E$, $B-A$, $B-B$, $B-C$, $B-D$, $B-E$ $C-A$, $C-B$, $C-C$, $C-D$, $C-E$ $D-A$, $D-B$, $D-C$, $D-D$, $D-E$ $E-A$, $E-B$, $E-C$, $E-D$, or $E-E$?

Simplify the following where possible:

71. $\begin{bmatrix} -2 & -3 \\ -2 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

72. $\begin{bmatrix} 6 & -8 \\ 9 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

73. $\begin{bmatrix} 8 & 7 & -8 \\ -7 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -7 & -1 \\ 1 & -8 & -8 \end{bmatrix}$

74. $\begin{bmatrix} 1 & -1 & -3 \\ 6 & 8 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 9 \\ -9 & 1 \\ 8 & 8 \end{bmatrix}$

75. $\begin{bmatrix} -5 & 2 & -9 & 2 \end{bmatrix} + \begin{bmatrix} -7 & 2 & 0 & -7 \end{bmatrix}$

76. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

77. $\begin{bmatrix} -6 & -2 & 8 & 4 \\ -1 & -1 & 5 & 10 \end{bmatrix} + \begin{bmatrix} 4 & -7 & 0 \\ 5 & -2 & 8 \end{bmatrix}$

78. $\begin{bmatrix} 8 & 8 & 8 \\ 7 & 7 & 7 \\ 3 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$

79. $\begin{bmatrix} -10 & -7 & 3 & 10 & 7 \\ 9 & -6 & -1 & 8 & 1 \\ 1 & -9 & 7 & -6 & 2 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 5 & 3 & 1 \\ -4 & -5 & 7 & 6 & -10 \\ 3 & 9 & -9 & 8 & 4 \end{bmatrix}$

80. $\begin{bmatrix} -9 & -8 & 7 & 0 \\ 8 & 2 & 8 & 2 \\ 2 & 8 & -7 & 10 \end{bmatrix} - \begin{bmatrix} -6 & -10 & -7 & 0 \\ -2 & -8 & -3 & 6 \\ 8 & -10 & -9 & 10 \end{bmatrix}$

81. $\begin{bmatrix} 6 & -2 & -7 & 2 \\ -4 & -8 & 9 & -2 \end{bmatrix} - \begin{bmatrix} -10 & -3 & -9 & -5 \\ -1 & -1 & -3 & 5 \end{bmatrix}$

82. $\begin{bmatrix} -6 & 3 & 5 & -10 & -2 \\ 4 & 5 & -8 & 2 & 1 \\ 0 & 1 & 5 & -7 & 2 \end{bmatrix} - \begin{bmatrix} -3 & -6 & -2 & -10 & 0 \\ 8 & -5 & 9 & 9 & 4 \\ 9 & 0 & -2 & 0 & -5 \end{bmatrix}$

83. $\begin{bmatrix} -3 & 5 \\ 7 & 0 \end{bmatrix} - \begin{bmatrix} 7 & -2 \\ 10 & 10 \end{bmatrix}$

84. $\begin{bmatrix} 1 & 1 & -6 & 0 & 10 \\ -8 & -6 & 7 & -1 & 8 \\ 4 & 9 & -4 & 6 & -1 \end{bmatrix} - \begin{bmatrix} -7 & -2 & 3 & -5 & -4 \\ 9 & -2 & -3 & -8 & 1 \\ -5 & 2 & 2 & 0 & -3 \end{bmatrix}$

85. $\begin{bmatrix} -6.9 & -1.8 \end{bmatrix} + \begin{bmatrix} -0.9 & -6.7 \end{bmatrix}$

86. $\begin{bmatrix} 6 & 6.3 \\ -8.1 & 6.8 \end{bmatrix} - \begin{bmatrix} -6 & 2.4 \\ -2.8 & 3 \end{bmatrix}$

87. $\begin{bmatrix} 2.8 & 5.6 \\ 0.5 & -9.4 \\ -4.3 & 9.3 \\ 6.9 & 1.4 \end{bmatrix} - \begin{bmatrix} 6 & -5.4 \\ 9.9 & 2.7 \end{bmatrix}$

88. $\begin{bmatrix} -2.2 & -4.1 \\ 9.8 & 8.9 \\ -8.7 & 10 \\ 7 & 9.6 \end{bmatrix} + \begin{bmatrix} 4.7 & 1.1 \\ 8.2 & -6.9 \\ 1.5 & 3.4 \\ 6.5 & -10 \end{bmatrix}$

89. $\begin{bmatrix} -5.3 & 2.7 & -3.4 & 3.3 \end{bmatrix} + \begin{bmatrix} -0.5 & 1.9 & -6 & 8.2 \end{bmatrix}$

90. $\begin{bmatrix} 3 & 3.5 & 5.7 \\ -6.5 & 1.7 & 9.6 \\ 8.6 & 6.3 & -8.7 \\ 5.8 & 9 & 8.2 \end{bmatrix} + \begin{bmatrix} -4.8 & 8 & 2.5 \\ -8.3 & -1.9 & -7.5 \\ -8 & 5.5 & 1.8 \\ 3 & 8.2 & 8.3 \end{bmatrix}$

Solve for the unknowns.

91. $\begin{bmatrix} a & 3 & -9.9 \\ 1.2 & 9.5 & -4.3 \end{bmatrix} - \begin{bmatrix} 3.8 & -2.5 & -7.7 \\ -10 & b & -8.3 \end{bmatrix} = \begin{bmatrix} -5.9 & 5.5 & c \\ 11.2 & 8.4 & 4 \end{bmatrix}$

92. $\begin{bmatrix} a & -3.2 \end{bmatrix} + \begin{bmatrix} -7.1 & b \end{bmatrix} = \begin{bmatrix} -5.3 & -10.3 \end{bmatrix}$

93. $\begin{bmatrix} -4.9 & a \\ 10 & -7.1 \end{bmatrix} + \begin{bmatrix} 7.5 & 0.9 \\ 6.5 & b \end{bmatrix} = \begin{bmatrix} 2.6 & 3.3 \\ c & -5.1 \end{bmatrix}$

94. $\begin{bmatrix} -4.9 & a & 2.3 & 7.3 \end{bmatrix} + \begin{bmatrix} d & 3.1 & b & -3 \end{bmatrix} = \begin{bmatrix} -1.2 & 3.2 & -2.4 & c \end{bmatrix}$

95. Given $A + I = \begin{bmatrix} 2.5 & -1.3 & -5.7 \\ -8.2 & -7.6 & 4.7 \\ 0.2 & -5.1 & 1.6 \end{bmatrix}$, determine A.

96. Given $B - I = \begin{bmatrix} 1.3 & 7.3 & 6.5 & 5 \\ -3.1 & -4.1 & 4 & -9.6 \\ 1.9 & 8.5 & 6 & 3.6 \\ -7.2 & 0.4 & -7.7 & 8.7 \end{bmatrix}$ determine B.

97. Given $I - C = \begin{bmatrix} -1 & -1 & 0 \\ -2 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ determine C

98. Given $I - D = \begin{bmatrix} 1 & -2 & 2 & -1 \\ 2 & 2 & -2 & 2 \\ -1 & 2 & -2 & -2 \\ 2 & 0 & 1 & 0 \end{bmatrix}$, determine D.

1.7 Scalar Multiplication

Simplify:

99. $-5\begin{bmatrix} -1 & -4 \\ -4 & 3 \\ -3 & 5 \\ -5 & 5 \end{bmatrix}$

100. $-1\begin{bmatrix} -2 & 1 & 5 & 5 \\ 4 & 1 & 4 & -1 \\ 2 & 3 & -5 & -2 \\ -4 & 5 & -5 & 4 \end{bmatrix}$

101. $4\begin{bmatrix} -1 & -2 & 4 \\ 4 & 3 & 4 \\ -2 & 4 & 4 \\ -5 & 5 & -2 \end{bmatrix}$

102. $3\begin{bmatrix} -5 \\ -2 \\ -2 \end{bmatrix}$

103. $\begin{bmatrix} -3 & -3 & 5 \\ 0 & 1 & 2 \\ 5 & 3 & -4 \end{bmatrix} + 3I$

104. $\begin{bmatrix} -3 & 0 & -2 \\ 5 & 1 & 5 \\ 0 & -3 & 4 \end{bmatrix} - 4I$

105. $2\begin{bmatrix} 2 & 2 & -5 \end{bmatrix} + 2\begin{bmatrix} 3 & -2 & -2 \end{bmatrix}$

106. $-5\begin{bmatrix} 4 & 1 \\ 3 & -1 \\ 1 & -3 \end{bmatrix} + 3\begin{bmatrix} 3 & -3 \\ 0 & -5 \\ 5 & 1 \end{bmatrix}$

107. $4\begin{bmatrix} -2 & 2 \\ -1 & -3 \\ -4 & 0 \end{bmatrix} + 3\begin{bmatrix} -4 & 2 \\ -2 & 0 \\ 1 & 1 \end{bmatrix}$

108. $3\begin{bmatrix} 1 & -5 & 3 \\ 1 & -1 & -5 \\ -3 & 5 & 3 \\ -2 & 1 & 3 \end{bmatrix} - 3\begin{bmatrix} -1 & 0 & 5 \\ 2 & -2 & -5 \\ -1 & -1 & -5 \\ -4 & 1 & -5 \end{bmatrix}$

Solve for the unknowns.

109. $-1\begin{bmatrix} a & 4 & 1 & -4 \\ 5 & 1 & 0 & e \end{bmatrix} + 4\begin{bmatrix} -4 & b & 0 & -1 \\ 1 & 2 & c & -3 \end{bmatrix} = \begin{bmatrix} -21 & -16 & -1 & f \\ d & 7 & -8 & -12 \end{bmatrix}$

110. $-2\begin{bmatrix} -2 & a & 1 \\ b & -2 & -3 \\ -4 & c & -5 \end{bmatrix} - 3\begin{bmatrix} d & 1 & -3 \\ 1 & 1 & -5 \\ 2 & 5 & e \end{bmatrix} = \begin{bmatrix} -2 & -3 & 7 \\ -5 & f & 21 \\ 2 & -23 & 10 \end{bmatrix}$

111. $a\begin{bmatrix} 5 \\ d \\ -1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} b \\ -5 \\ c \\ 0 \end{bmatrix} = \begin{bmatrix} -11 \\ -14 \\ 23 \\ -6 \end{bmatrix}$

112. $a\begin{bmatrix} c & 0 & 1 & 3 \\ 1 & d & -5 & 3 \end{bmatrix} + b\begin{bmatrix} 1 & -3 & e & 1 \\ -3 & 0 & 2 & f \end{bmatrix} = \begin{bmatrix} -13 & 15 & -13 & g \\ 17 & 0 & -20 & 21 \end{bmatrix}$

1.8 Matrix Multiplication

For each of the following products, determine if the product is possible, and if so give the size of the product.

113. A (6×6) matrix times a (6×4) matrix

114. A (4×1) matrix times a (1×7) matrix

115. A (4×5) matrix times a (5×1) matrix

116. A (6×2) matrix times a (2×2) matrix

117. A (2×2) matrix times a (2×6) matrix

118. A (5×1) matrix times a (1×4) matrix

119. A (6×4) matrix times a (4×6) matrix

120. A (8×1) matrix times a (5×2) matrix

121. A (1×1) matrix times a (1×4) matrix

122. A (6×8) matrix times a (8×5) matrix

123. A (2×6) matrix times a (5×6) matrix

124. A (8×1) matrix times a (6×8) matrix

For the following, give whichever of the products AB and BA are possible.

125. $A = \begin{bmatrix} 4 & 2 & -2 \\ 5 & -5 & 3 \\ 4 & 4 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 & 5 \\ -2 & -5 & 2 \\ -1 & 0 & 5 \end{bmatrix}$

126. $A = \begin{bmatrix} 5 & 2 \\ 0 & 5 \\ -5 & 3 \\ -3 & -1 \\ -1 & -3 \end{bmatrix}$ $B = \begin{bmatrix} -4 \\ -5 \end{bmatrix}$

127. $A = \begin{bmatrix} -4 \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$

128. $A = \begin{bmatrix} -4 & -1 & 3 & 5 & 0 \\ 0 & -2 & 1 & -1 & 2 \\ -3 & -3 & -5 & -2 & 5 \\ 5 & 0 & -3 & -4 & -5 \\ -3 & -2 & -5 & -2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 3 & 4 & -2 & 4 \\ -5 & 5 & -3 & 2 & -4 \\ -5 & -2 & 0 & -4 & 5 \\ 1 & -3 & 1 & 2 & 5 \end{bmatrix}$

129. $A = \begin{bmatrix} 3 \end{bmatrix}$ $B = \begin{bmatrix} -4 & 2 \end{bmatrix}$

130. $A = \begin{bmatrix} 0 & 1 \\ -5 & -3 \\ -4 & 3 \\ -3 & -3 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -4 & -2 & 1 & 4 & 3 \\ -5 & -3 & -1 & -4 & -1 \\ 0 & 0 & 5 & 0 & 4 \\ 2 & -4 & -2 & -1 & 2 \\ -3 & -3 & -3 & 1 & 2 \end{bmatrix}$

131. $A = \begin{bmatrix} -3 & -3 & 0 & -1 \\ -4 & 5 & -2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & -2 \\ -2 & 4 \\ 0 & 3 \\ -4 & 2 \end{bmatrix}$

132. $A = \begin{bmatrix} 4 & -2 \\ -5 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -5 & 4 & 0 & 2 \\ -3 & -1 & 4 & -3 \end{bmatrix}$

133. For $A = \begin{bmatrix} 1 & -1 \\ -4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 \\ -1 & 2 \end{bmatrix}$, show that $AB + A = A(B + I)$.

134. For $C = \begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & -4 \\ -4 & 4 \end{bmatrix}$, show that $CD + D = (C + I)D$.

135. For $E = \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}$ and $F = \begin{bmatrix} -4 & 2 \\ 4 & -3 \end{bmatrix}$ show that $EF + 3E = E(F + 3I)$.

136. For $G = \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix}$ and $H = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix}$ show that $5H - GH = (5I - G)H$

137. For $J = \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$, $K = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ and $L = \begin{bmatrix} -1 & -4 \\ 3 & -2 \end{bmatrix}$ show that $(JK)L = J(KL)$.

138. For $M = \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix}$, $N = \begin{bmatrix} 5 & 0 \\ 4 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} -3 & -5 \\ 1 & 4 \end{bmatrix}$ show that $(MN)P = M(NP)$.

139. Using appropriate technology, calculate $\begin{bmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{bmatrix}^5$

140. Using appropriate technology, calculate $\begin{bmatrix} 0.3 & 1.2 \\ 0.8 & 1.3 \end{bmatrix}^6$

141. Using appropriate technology, calculate $\begin{bmatrix} 0 & 0.7 & 0.9 \\ 0.4 & 0 & 0.1 \\ 0.9 & 0.3 & 0.5 \end{bmatrix}^8$

142. Using appropriate technology, calculate $\begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.3 & 1.4 & 0.8 \\ 1.7 & 1.3 & 2 \end{bmatrix}^4$

143. For $Q = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$ show that $(Q + I)^2 = Q^2 + 2Q + I$

144. For $R = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ show that $R^2 - I = (R + I)(R - I)$

1.9 DETERMINANT

145. Calculate $\det\left(\begin{bmatrix} -6 & 0 \\ 4 & -1 \end{bmatrix}\right)$

146. Calculate $\det\left(\begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix}\right)$

147. Calculate $\det\left(\begin{bmatrix} 8 & -3 \\ -9 & -2 \end{bmatrix}\right)$

148. Calculate $\det\left(\begin{bmatrix} -5 & 8 \\ 9 & 8 \end{bmatrix}\right)$

149. Calculate $\begin{vmatrix} -4 & 5 \\ 2 & 3 \end{vmatrix}$

150. Calculate $\begin{vmatrix} -2 & 7 \\ 6 & 4 \end{vmatrix}$

151. Calculate $\begin{vmatrix} -9 & -6 \\ -8 & -8 \end{vmatrix}$

152. Calculate $\begin{vmatrix} -7 & -4 \\ -1 & -4 \end{vmatrix}$

153. For $A = \begin{bmatrix} 6 & -3 \\ 4 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -6 \\ 4 & 2 \end{bmatrix}$ show that $\det(AB) = \det(A)\det(B)$.

154. For $C = \begin{bmatrix} 2 & 9 \\ -3 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} -8 & -6 \\ 5 & 6 \end{bmatrix}$ show that $\det(AB) = \det(BA)$.

155. For $E = \begin{bmatrix} 1 & -4 \\ 6 & 3 \end{bmatrix}$ show that $\det(3E) = 3^2 \det(E)$

156. For $F = \begin{bmatrix} -3 & -8 \\ 3 & 5 \end{bmatrix}$ show that $\det(10F) = 10^2 \det(F)$

157. Calculate $\det\left(\begin{bmatrix} -1 & 3 & 4 \\ -1 & 2 & 1 \\ -1 & -4 & 5 \end{bmatrix}\right)$

158. Calculate $\det\left(\begin{bmatrix} -2 & -4 & 5 \\ 3 & 3 & -1 \\ -3 & 1 & 0 \end{bmatrix}\right)$

159. Calculate $\begin{vmatrix} 3 & 2 & 5 \\ -1 & -4 & -5 \\ 0 & 3 & -4 \end{vmatrix}$

160. Calculate $\begin{vmatrix} 2 & -3 & 3 \\ 2 & -4 & -3 \\ 2 & -4 & 3 \end{vmatrix}$

161. $\begin{bmatrix} 3 & a \\ -5 & 5 \end{bmatrix}$ has determinant of -10. Determine a.

162. $\begin{bmatrix} -3 & -2 \\ -5 & b \end{bmatrix}$ has determinant of -4. Determine b.

163. $\begin{bmatrix} -9 & -2 \\ c & 10 \end{bmatrix}$ has determinant of -76. Determine c.

164. $\begin{bmatrix} d & -9 \\ -9 & -9 \end{bmatrix}$ has determinant of -54. Determine d.

165. $\det\left(\begin{bmatrix} e & 8 \\ -5 & -9 \end{bmatrix} + I\right) = 64$. Determine e.

166. $\det\left(\begin{bmatrix} 6 & f \\ 1 & 9 \end{bmatrix} + I\right) = 80$. Determine f.

167. $\det\left(\begin{bmatrix} g & 2 \\ 5 & g \end{bmatrix}\right) = 54$. Determine g.

168. $\det\left(\begin{bmatrix} -2 & h \\ h & -9 \end{bmatrix}\right) = 2$. Determine h.

169. $\det\left(\begin{bmatrix} k & 7 \\ k & k \end{bmatrix}\right) = 30$. Determine k.
170. $\det\left(\begin{bmatrix} m & m \\ m & 8 \end{bmatrix}\right) = -12$. Determine m.

1.10 Inverse Matrices

171. Which of these matrices is singluar? $A = \begin{bmatrix} 4 & 5 \\ 2 & 10 \end{bmatrix}, B = \begin{bmatrix} 0 & -7 \\ 0 & -4 \end{bmatrix}, C = \begin{bmatrix} 5 & -1 \\ 30 & 6 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ -1 & 0 \end{bmatrix}$
172. Which of these matrices is singluar? $A = \begin{bmatrix} 0 & 10 \\ 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix}, C = \begin{bmatrix} 9 & -3 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} -5 & 2 \\ 7 & -3 \end{bmatrix}$

For the following matrices, determine the inverse matrix where possible.
173. $\begin{bmatrix} 7 & -9 \\ -2 & -3 \end{bmatrix}$
174. $\begin{bmatrix} 10 & 2 \\ 10 & -10 \end{bmatrix}$
175. $\begin{bmatrix} -6 & 4 \\ 0 & 0 \end{bmatrix}$
176. $\begin{bmatrix} 9 & 5 \\ -1 & -9 \end{bmatrix}$
177. $\begin{bmatrix} -4 & -6 \\ 1 & 4 \end{bmatrix}$
178. $\begin{bmatrix} 4 & 4 \\ -7 & 6 \end{bmatrix}$
179. $\begin{bmatrix} 9 & -4 \\ -8 & -4 \end{bmatrix}$
180. $\begin{bmatrix} 2 & 4 \\ -8 & -6 \end{bmatrix}$
181. $\begin{bmatrix} -7 & -2 \\ 4 & 0 \end{bmatrix}$
182. $\begin{bmatrix} 6 & -7 \\ -2 & -1 \end{bmatrix}$
183. Given $M = \begin{bmatrix} 1 & -3 & 9 \\ 1 & -7 & -1 \\ 1 & -7 & -4 \end{bmatrix}$ follow these steps to find M^{-1}.

a) Create an augmented matrix with M on the left and a 3×3 identity matrix on the right.

b) Use row operations to transform the augmented matrix to row echelon form with leading non-zero matrix entries equal to 1.
c) Continue to use row operations to transform the left of the augmented matrix to a 3×3 identity matrix.
d) Confirm that the right of the augmented matrix is M^{-1} by multiplying by M.

184. Given $N = \begin{bmatrix} 1 & -1 & 1 \\ 8 & -9 & 9 \\ -5 & -4 & -7 \end{bmatrix}$ follow these steps to find N^{-1}.

a) Create an augmented matrix with N on the left and a 3×3 identity matrix on the right.
b) Use row operations to transform the augmented matrix to row echelon form with leading non-zero matrix entries equal to 1.
c) Continue to use row operations to transform the left of the augmented matrix to a 3×3 identity matrix.
d) Confirm that the right of the augmented matrix is N^{-1} by multiplying by N.

185. Use an appropriate manual technique or technology to find the inverse of $\begin{bmatrix} 1 & -5 & 3 \\ 2 & -9 & 7 \\ -1 & 6 & 0 \end{bmatrix}$

186. Use an appropriate manual technique or technology to find the inverse of $\begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 4 \\ 4 & 0 & 10 \end{bmatrix}$

187. Find the inverse of $\begin{bmatrix} 1 & -3 & -8 \\ 3 & 2 & -1 \\ 4 & -1 & -8 \end{bmatrix}$

188. Find the inverse of $\begin{bmatrix} -2 & -4 & 6 \\ -8 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix}$

189. Find the inverse of $\begin{bmatrix} -5 & -4 & 3 \\ -3 & -9 & -2 \\ 2 & -5 & -5 \end{bmatrix}$

190. Find the inverse of $\begin{bmatrix} 0 & -2 & -8 \\ 9 & 4 & 5 \\ 4 & 0 & -10 \end{bmatrix}$

191. Solve for P: $P\begin{bmatrix} 5 & -4 & -8 \\ 3 & 9 & -4 \\ -8 & -10 & -9 \end{bmatrix} = \begin{bmatrix} 1177 & 0 & 0 \\ 0 & 1177 & 0 \\ 0 & 0 & 1177 \end{bmatrix}$

192. Solve for Q: $Q\begin{bmatrix} 4 & 8 & 8 \\ 2 & 8 & 3 \\ -8 & -10 & -3 \end{bmatrix} = \begin{bmatrix} 116 & 0 & 0 \\ 0 & 116 & 0 \\ 0 & 0 & 116 \end{bmatrix}$

193. Solve for R: $\begin{bmatrix} 7 & -7 & 5 \\ 7 & 0 & -3 \\ -6 & -4 & 5 \end{bmatrix}R = 105I$

194. Solve for S: $\begin{bmatrix} -7 & 7 & 0 \\ -1 & 5 & 3 \\ 1 & -6 & 10 \end{bmatrix}S = 385I$

1.11 Solving Matrix Equations

For each of the following, $T = \begin{bmatrix} 6 & 10 \\ 4 & -5 \end{bmatrix}, U = \begin{bmatrix} 4 & -3 \\ -2 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 7 & -4 \\ -8 & 5 \end{bmatrix}$.

195. Use matrix algebra to solve $X + 3I = T$

196. Use matrix algebra to solve $X + 5I = U$

197. Use matrix algebra to solve $2I - X = V$

198. Use matrix algebra to solve $4I - X = U$

199. Use matrix algebra to solve $UX = V$

200. Use matrix algebra to solve $VX = T$

201. Use matrix algebra to solve $XT + 3I = U$

202. Use matrix algebra to solve $XV - U = 5I$

203. Solve $XT + XU = 10I$

204. Solve $VX - UX = 25I$

205. Solve $5UX - 9X = T$

206. Solve $12X + 2VX = 7U$

207. Solve $UXU^{-1} = V$

208. Solve $2T^{-1}XT = 3U$

209. Solve $X = U + UX + U^2X$

210. Solve $X = TV - XV - XT$

211. Given $W = \left[\begin{smallmatrix} w & 2w-1 \\ 1 & -w \end{smallmatrix}\right]$, determine what value(s) of w make W its own inverse.

212. Given $A = \left[\begin{smallmatrix} a & 5-3a \\ 2 & -a \end{smallmatrix}\right]$, determine what value(s) of a result in $A^2 = 2I$.

213. Given $B = \left[\begin{smallmatrix} b-1 & 3b \\ 1 & 5-2b \end{smallmatrix}\right]$, determine what values of b result in $\det(B) = \det\bigl(B^{-1}\bigr)$.

214. Given $C = \left[\begin{smallmatrix} 0 & 2c+1 \\ c-1 & c^2-5 \end{smallmatrix}\right]$, determine what value(s) of c result in $\det(C) = 4\det\bigl(C^{-1}\bigr)$.

2 Matrix Applications

2.1 Solving Systems of Linear Equations

Represent the following systems of simultaneous equations as augmented matrices, then use row operations to solve the system.

1. $3x + 2y = 5$

$x - 4y = 11$

2. $-4x - 3y = -58$

$2x + 5y = 64$

3. $-7x - 6y = 13$

$-4x - 3y = 10$

4. $-8x - 9y = 0$

$2x - 5y = -58$

5.
$$\begin{aligned} 9x - y &= 6 \\ -7x - 4y &= 8 \end{aligned}$$

6.
$$\begin{aligned} -8x - 3y &= 3 \\ 8x - 7y &= 9 \end{aligned}$$

7.
$$\begin{aligned} 7x + 5z &= 48 \\ 5 + 6y + 6z &= 33 \\ -5x - 5y - 2z &= -44 \end{aligned}$$

8.
$$\begin{aligned} -3x - 10y + 5z &= -7 \\ 7x - y &= 30 \\ 7x + y + z &= 23 \end{aligned}$$

9.
$$\begin{aligned} -6x + 7y + 8z &= -104 \\ 10x + 10y - z &= -111 \\ -9x - 7y + 6z &= 38 \end{aligned}$$

10.
$$\begin{aligned} -3x + 5y - 10z &= 114 \\ -3x + 5y + 5z &= -36 \\ 10x + 6y + 8z &= -10 \end{aligned}$$

11.
$$\begin{aligned} -2x - 5y &= 0 \\ 3x + 4y &= 10 + 7z \\ -8x &= 9y + 10z - 4 \end{aligned}$$

12.
$$\begin{aligned} 4z &= 8y + 5 \\ 8y + 2 &= 6x + 8z \\ 10z + 2 &= 4x + 8y \end{aligned}$$

13.
$$\begin{aligned} w - 6x + 9y + 5z &= 11 \\ -4w + 2x - 4y + 6z &= 10 \\ 8w - x - y - 3z &= 7 \\ -3w + 6x - 10y &= -1 \end{aligned}$$

14.
$$\begin{aligned} -6w + 3x - 3y - 7z &= 61 \\ 2w + 6y + 7z &= -58 \\ 2w + 6x + 10y + 9z &= -128 \\ 2w + 2x + 5y + 9z &= -82 \end{aligned}$$

15.
$$\begin{aligned} 9w + 8x - y - 10z &= -3 \\ 5w + 5x + 4y + 4z &= -9 \\ -9w - 3x - 2z &= 7 \\ 2w + 4x + 8y - z &= 4 \end{aligned}$$

16.
$$\begin{aligned} 8w - 10x - 3y + 4z &= -7 \\ 10w + 10x + 10y + 3z &= -7 \\ -5w + 2x - 6y - 4z &= -2 \\ 2w + 9x - 5y - 10z &= -7 \end{aligned}$$

17.
$$\begin{aligned} 6v - 9w + 5y - 5z &= -10 \\ 3w - 2x + y - 9z &= -5 \\ -9v + 4w + 9x + 5z &= 9 \\ -v - w + 7x - 8y + 9z &= 6 \\ 4v + 3w - 14x + 2y &= -2 \end{aligned}$$

18.
$$\begin{aligned} -2v - 4w - 6x + 4y + 10z &= 2 \\ -7v + 5w + 10x - 5y + 9z &= -7 \\ 2v - 10x + 9y + 6z &= 0 \\ 4v + 9w + 9x + 3y - 8z &= 10 \\ 2w + 5x - 2y + 4z &= 2 \end{aligned}$$

19. The following augmented matrix represents a system of linear equations. What must the value of a be if the system has no solution?

$$\left[\begin{array}{cc|c} 4 & 3 & -9 \\ a & -7 & -2 \end{array}\right]$$

20. The following augmented matrix represents a system of linear equations. What must the value of a be if the system has no solution?

$$\left[\begin{array}{cc|c} 2 & a & 2 \\ 7 & 7 & -1 \end{array}\right]$$

21. The following augmented matrix represents a system of linear equations. What must the value of a be if the system has no solution?

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 10 \\ -3 & -6 & a & 6 \\ -5 & 2 & 3 & 9 \end{array}\right]$$

22. The following augmented matrix represents a system of linear equations. What must the value of a be if the system has no solution?

$$\left[\begin{array}{ccc|c} 2 & 3 & a & 3 \\ -3 & 1 & 10 & 6 \\ 1 & -4 & -9 & 3 \end{array}\right]$$

23. The following augmented matrix represents a system of linear equations with an infinite number of solutions. Determine the value of a and b.

$$\left[\begin{array}{cc|c} 10 & 6 & b \\ -10 & a & -8 \end{array}\right]$$

24. The following augmented matrix represents a system of linear equations with an infinite number of solutions. Determine the value of a and b.

$$\left[\begin{array}{cc|c} 5 & 1 & -10 \\ 1 & a & b \end{array}\right]$$

25. The following augmented matrix represents a system of linear equations with an infinite number of solutions. Determine the value of a and b.

$$\left[\begin{array}{cc|c} a & -5 & -6 \\ -6 & b & -10 \end{array}\right]$$

26. The following augmented matrix represents a system of linear equations with an infinite number of solutions. Determine the value of a and b.

$$\left[\begin{array}{cc|c} 9 & a & 9 \\ b & 1 & 5 \end{array}\right]$$

27. The following augmented matrix represents a system of linear equations with an infinite number of solutions. Determine the value of a and b.

$$\left[\begin{array}{ccc|c} -3 & -3 & 6 & b \\ -10 & 17 & -10 & 6 \\ -1 & 2 & a & -5 \end{array}\right]$$

28. The following augmented matrix represents a system of linear equations with an infinite number of solutions. Determine the value of a and b.

$$\left[\begin{array}{ccc|c} 8 & 5 & 4 & 2 \\ 40 & 20 & 60 & b \\ 0 & a & 40 & 53 \end{array}\right]$$

Rewrite each the following matrix equations as a systems of linear equations.

29. $\left[\begin{smallmatrix} -1 & 7 \\ 4 & 3 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} -3 \\ 7 \end{smallmatrix}\right]$

30. $\left[\begin{smallmatrix} -3 & 3 \\ -2 & 4 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} -7 \\ 10 \end{smallmatrix}\right]$

31. $\begin{bmatrix} 0 & 6 & -5 \\ -5 & -9 & -10 \\ -1 & -5 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$

32. $\begin{bmatrix} 0 & -4 & 10 \\ -9 & -5 & -5 \\ -10 & -6 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -10 \\ 0 \end{bmatrix}$

33. $\begin{bmatrix} 6 & -2 & -8 & -1 \\ -4 & -7 & -6 & 7 \\ -2 & -2 & 2 & -2 \\ -8 & -7 & 10 & -5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 1 \\ -8 \end{bmatrix}$

34. $\begin{bmatrix} -1 & 9 & 6 & -3 \\ 2 & -2 & 7 & -10 \\ -1 & 9 & 2 & -10 \\ -6 & -7 & 7 & -6 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -4 \\ -8 \end{bmatrix}$

35. $\begin{bmatrix} 9 & 9 & -6 & -6 & -7 \\ -2 & 3 & -1 & -5 & -5 \\ -10 & -3 & -2 & -6 & 4 \\ 6 & -2 & 5 & 7 & 3 \\ -6 & -10 & 6 & 8 & -9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -65 \\ 49 \\ 77 \\ -48 \\ 97 \end{bmatrix}$

36. $\begin{bmatrix} -2 & 10 & -10 & -8 & -10 \\ -3 & 6 & 2 & -6 & 6 \\ -6 & 1 & -9 & 0 & -6 \\ -6 & -4 & 7 & -3 & 0 \\ -8 & -8 & -9 & 6 & 10 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 85 \\ 90 \\ 30 \\ 9 \\ 110 \end{bmatrix}$

Write each the following systems of equations as a single matrix equation of the form $AX = B$ where A is a matrix of coefficients, X is a column matrix of unknowns, and B is a column matrix of constants.

37. $\begin{aligned} 3x + 4y &= -5 \\ 3y &= 5 \end{aligned}$

38. $\begin{aligned} -3x + 3y &= 6 \\ -10x &= -2 \end{aligned}$

39. $\begin{aligned} -3x - y - 9z &= 0 \\ 3x - 8y + 3z &= 0 \\ 9x - 7y + 9z &= 10 \end{aligned}$

40.
$$\begin{aligned} -4y + 10z &= 0 \\ -9x + 3y + 6z &= -6 \\ -10x - 3y &= 7 \end{aligned}$$

41.
$$\begin{aligned} -6w + 3x - 2y + 2z &= -9 \\ -7w + 10x + 10z &= 9 \\ -3w - 2x + 2y + 5z &= 4 \\ -9w + 9x - 3y - 5z &= 7 \end{aligned}$$

42.
$$\begin{aligned} 9w + 3x - 3y - 9z &= 6 \\ -6w + 5x - 7y + 8z &= -1 \\ 8w + 9x - 6y - 10z &= 8 \\ 2w + 7x - 5y - 2z &= 7 \end{aligned}$$

43.
$$\begin{aligned} w + 9y &= 23 \\ 9v + 5x + 7z &= 41 \\ 8v + 10w &= 8 \\ 10v + 3y + 8z &= -58 \\ 4v + x + 2y &= 4 \end{aligned}$$

44.
$$\begin{aligned} x + 7y &= 107 \\ 3v + 9y &= 97 \\ 8v + 8w + 9x + y &= 47 \\ 3v + 2z &= 41 \\ 5y &= 37 \end{aligned}$$

Use inverse matrices to solve the following systems of equations.

45. $-3x - 2y = -13$

$9x - y = 4$

46. $-9x + 2y = 17$

$3x + 8y = 29$

47. $x + 8y = -72$

$-3x - 6y = 36$

48. $-7x + y = -25$

$3x + 10y = 85$

49. $3x - y = 19$

$-2x + 8y = 2$

50. $-7x + y = -25$

$3x + 10y = 85$

51. $-9.8x + 9.1y = 52$

$8.4x + 5.6y = 58$

52. $-9.7x - 6.1y = 48$

$6.1x + 1.5y = -70$

53. Use the matrix equation

$$\begin{bmatrix} 1 & 9 & -4 \\ -5 & -3 & -1 \\ 2 & 8 & -2 \end{bmatrix}\begin{bmatrix} 14 & -14 & -21 \\ -12 & 6 & 21 \\ -34 & 10 & 42 \end{bmatrix} = \begin{bmatrix} 42 & 0 & 0 \\ 0 & 42 & 0 \\ 0 & 0 & 42 \end{bmatrix}$$

to solve the system of equations

$$\begin{aligned} 14x - 14y - 21z &= -315 \\ -12x + 6y + 21z &= 273 \\ -34x + 10y + 42z &= 546 \end{aligned}$$

54. Use the matrix equation $\begin{bmatrix} 18 & -2 & -36 \\ -58 & 7 & 117 \\ 16 & -2 & -32 \end{bmatrix}\begin{bmatrix} 5 & 4 & 9 \\ 8 & 0 & -9 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ to solve the system of equations

$$\begin{aligned} 18x - 2y - 36z &= 3 \\ -58x + 7y + 117z &= 1 \\ 16x - 2y - 32z &= 4 \end{aligned}$$

55. Use the matrix equation

$$\begin{bmatrix} 4 & 8 & -16 & -7 \\ -4 & -1 & 2 & 0 \\ 1 & 2 & -4 & 0 \\ -6 & -12 & 17 & 7 \end{bmatrix}\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & -1 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & -4 & 0 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$

to solve the system of equations

$$\begin{aligned} 2x + y &= 3 \\ 2w - x + 2z &= 4 \\ w + 2y + z &= 1 \\ w - 4y &= 9 \end{aligned}$$

56. Use the matrix equation

$$\begin{bmatrix} -1 & 1 & 5 & 2 \\ 3 & 8 & -15 & -6 \\ 2 & 9 & -21 & -4 \\ -3 & 14 & -29 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 3 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & -3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

to solve the system of equations

$$\begin{aligned} w + 3y - 2z &= 20 \\ 3w + x &= 5 \\ w + x - y &= 9 \\ 2w - 3x + 4y - z &= 7 \end{aligned}$$

57. Use the matrix equation

$$\begin{bmatrix} 8 & 1 & 4 & 7 & -3 \\ 15 & 3 & 12 & -6 & 0 \\ 14 & 4 & 16 & 1 & -3 \\ 7 & 2 & 8 & -4 & 3 \\ 9 & 0 & 9 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 2 & -4 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

to solve the system of equations

$$\begin{aligned} v + w &= 5 + x \\ v + x + 2y &= 8 + 4z \\ x + z &= v + w - 1 \\ v + y &= w + 7 \\ v + 3y &= w + x + 1 \end{aligned}$$

58. Use the matrix equation

$$\begin{bmatrix} 24 & -42 & -18 & 12 & 6 \\ 15 & -24 & -9 & 6 & 3 \\ -4 & 4 & 2 & 0 & 2 \\ 10 & -22 & -8 & 6 & 4 \\ 3 & -12 & -3 & 6 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 & -2 & 0 \\ 1 & 0 & 1 & -2 & 0 \\ -2 & 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & -3 & 2 \\ 0 & 1 & 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

to solve the system of equations

$$\begin{aligned} 2w + x &= 2y + 2 \\ 2y + 1 &= v + x \\ 3w + z + 2 &= 2v \\ v + 2z &= 3 + 3y \\ w + 3x &= z + 1 \end{aligned}$$

59. N is a two-digit number that is one more than thirteen times the difference between the digits. N is also four more than eight times the sum of the digits. Write this information as a system of linear equations then solve the system using an inverse matrix.

60. N is a two-digit number where N is five more than eight times the sum of its digits and when we reverse the digits we get a number equal to $N-45$. Write this information as a system of linear equations then solve the system using an inverse matrix.

61. N is a two-digit number where the sum of the digits is one less than a third of N and N is 54 less than the number obtained by reversing its digits. Write this information as a system of linear equations then solve the system using matrices.

62. N is a two-digit number that is equal to seven times the sum of its digits and where the tens digit is double the ones digit. Write this information as a system of linear equations then solve the system using matrices.

63. N is a three-digit number. N is eleven more than 21 times the sum of its digits. When we reverse the digits of N we get a number that is 495 larger than N. The tens digit of N is double the hundreds digit. Write this information as a system of linear equations then solve the system using row operations or with an inverse matrix obtained using appropriate technology.

64. N is a three-digit number. $N + 7$ is equal to 23 times the sum of N's digits, but if we reverse N's digits we get a number equal to two more than 45 times the sum of the digits. The sum of the tens and units digits is equal to four and a half times the hundreds digit. Write this information as a system of linear equations then solve the system.

2.2 General Matrix Word Problems

65. A hat seller has stock in each of small, medium and large sizes for akubras, berets, caps and deer stalkers. The current stock on hand is

Style	Small	Medium	Large
Akubra	6	28	27
Beret	18	7	23
Caps	8	18	6
Deer stalker	0	4	9

The stock is valued at \$25 for each akubra, \$18 for each beret, \$3.50 for each cap and \$49 for each deer stalker.

a) Use matrix multiplication to determine the total value of stock on hand.

b) Large sizes have been overstocked and need to be discounted. Use a matrix multiplication to recalculate the total value of stock on hand with a 30% discount applied to all large sizes.

c) The selling price for each item includes a 50% mark-up and for every had sold, the sales staff member making the sale gets a \$1.20 bonus. Write a single matrix expression equation and use it to calculate the total revenue for the company if all the hats are sold. (Include the discount for large sizes.)

66. A clothing store sells shirts in four sizes (S, M, L and XL) and three styles (Tees, Polos and Dress shirts). The current stock on hand is

Style	S	M	L	XL
Tee	5	9	20	15
Polo	8	0	23	22
Dress	12	26	15	12

The stock is valued at \$9 for each tee, \$13 for each polo and \$28 for each dress shirt.

a) Use matrix multiplication to determine the total value of stock on hand.

b) XL and L sized shirts have been overstocked and need to be discounted. Use a matrix multiplication to recalculate the total value of stock on hand with a 10% discount applied L and 50% discount applied to XL.

c) The selling price for each item includes an 80% mark-up and for every had sold, the sales staff member making the sale gets a \$1.90 bonus. Write a single matrix expression equation and use it to calculate the total revenue for the store if all the shirts are sold. (Include the discount for large sizes.)

67. The top of the AFL ladder contains this information about games won, drawn and lost, and points for and against:

Club	W	D	L	PF	PA
CARL	17	2	-	2022	1437
GEEL	14	5	-	2194	1647
ESS	13	4	2	2162	1632
RICH	13	5	1	1781	1642
WCE	12	7	-	1760	1454
NMFC	12	7	-	1891	1722
WB	10	8	1	1591	1696
MELB	9	10	-	1659	1565

Represent this information in an 8×5 matrix then,

a) Show how a single matrix multiplication can be used to determine the number of games each club has played.

b) Four points are allocated for each game won, and two points for each tie. Show how a single matrix multiplication can be used to calculate points for each club.

c) Show how a single matrix multiplication can be used to calculate a “point difference” for each club, being Points For minus Points Against.

68. Singapore's professional football league shows the following results ladder with games won, drawn and lost, goals for and goals against:

Club	W	D	L	GF	GA
Warriors FC	8	0	3	22	18
Albirex Niigata FC (S)	7	2	3	15	8
Brunei DPMM FC	6	1	2	17	10
Balestier Khalsa FC	4	2	4	15	13
Tampines Rovers FC	4	2	4	12	11
Home United FC	3	4	5	14	15
Geylang International FC	3	3	5	12	13
Harimau Muda	3	1	3	9	11
Hougang United FC	1	4	6	9	18
Courts Young Lions	1	1	5	4	12

Represent this information in a 10×5 matrix then,

a) Show how a single matrix multiplication can be used to determine the number of games each club has played.

b) Three points are allocated for each game won, and one point for each draw. Show how a single matrix multiplication can be used to calculate points for each club.

c) Show how a single matrix multiplication can be used to calculate a goal difference for each club, being Goals For minus Goals Against.

69. Graham's Fertilizers uses raw ingredients with different N, P and K content:

Ingredient	N%	P%	K%	\$/kg
Bird guano	12	6.0	2.0	0.15
Phosphate rock	0	6.1	0	0.12
Wood ash	0	1.5	7.0	0.03

How much of each raw ingredient should Graham use to produce fertilizer containing 12kg of elemental Nitrogen (N), 10kg of Phosphorus (P) and 6kg of Potassium (K), and what will the ingredients cost?

70. A chemical analysis of oil from several varieties of olives shows varying percentages of the triglycerides identifed as OOO, POO, OOL and PLO as follows:

Variety	OOO	POO	OOL	PLO
A	43.3	25.2	10.9	5.4
B	43.3	25.7	10.2	4.8
C	38.6	28.4	10.7	6.2
D	38.7	22.9	14.2	6.6

Another sample is believed to be a blend of these four varieties, but the proportions of each in the blend is unknown. Chemical analysis of this sample yields

Variety	OOO	POO	OOL	PLO
X	41.9	25.3	11.2	5.4

Use matrices to determine the proportions of each variety in the blend.

71. A brand of Almond, Brazil and Cashew spread lists the following nutritional information (per 100g):

Calories	605
Total fat	53g
Protein	18g

Research about the constituent nuts reveals that they have the following nutritional contents (per 100g):

	Almond	Brazil	Cashew
Calories	567	656	588
Total fat	49	66	44
Protein	21	14	18

Determine the proportions that make up the Almond, Brazil and Cashew spread.

72. Peter wants to maintain the Carbon:Nitrogen ratio in his compost heap at 28:1 for optimum composting. He has just received 50kg of shredded newspaper to compost, with a C:N ratio of 175:1. Use matrices to determine how much chicken manure (with C:N ratio of 10:1) should he add to make a mixture with a C:N ratio of 28:1.

73. The curve $y = f(x)$ (where $f(x)$ is a quadratic equation) passes through the points $(-1, 5)$, $(1, 3)$ and $(2, -4)$. Determine $f(x)$.

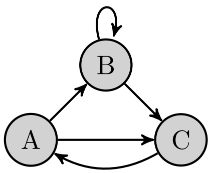
74. The curve $y = f(x)$ (where $f(x)$ is a quadratic equation) passes through the points $(-2, -10)$, $(1, -2)$ and $(3, -5)$. Determine $f(x)$.

75. The curve $y = f(x)$ (where $f(x)$ is a cubic equation) passes through the points $(-2, 5)$, $(-1, -2)$, $(1, -8)$ and $(3, -7)$. Determine $f(x)$.

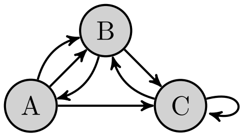
76. The curve $y = f(x)$ (where $f(x)$ is a cubic equation) passes through the points $(-2, 7)$, $(1, -4)$, $(2, 0)$ and $(3, -9)$. Determine $f(x)$.

2.3 Networks

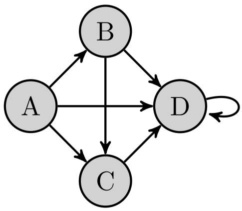
77. Write a matrix that represents the number of ways to travel from node to node in the following network:



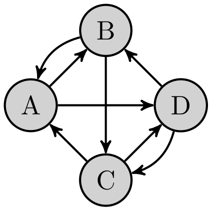
78. Write a matrix that represents the number of ways to travel from node to node in the following network:



79. Write a matrix that represents the number of ways to travel from node to node in the following network:



80. Write a matrix that represents the number of ways to travel from node to node in the following network:



81. The following matrix represents the number of ways to travel from node to node in a network. Draw the network.

$$\begin{array}{cc} & \textit{From:} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \textit{To:} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

82. The following matrix represents the number of ways to travel from node to node in a network. Draw the network.

$$\begin{array}{cc} & \textit{From:} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \textit{To:} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

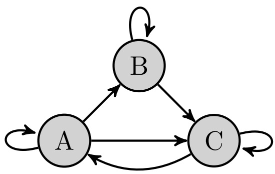
83. The following matrix represents the number of ways to travel from node to node in a network. Draw the network.

$$\begin{array}{cc} & \textit{From:} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \textit{To:} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

84. The following matrix represents the number of ways to travel from node to node in a network. Draw the network.

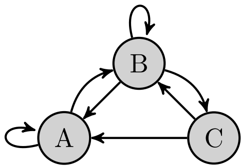
$$\begin{array}{ccc} & & \textit{From:} \\ & & \begin{array}{cccc} A & B & C & D \end{array} \\ \textit{To:} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

85. Consider the following network:



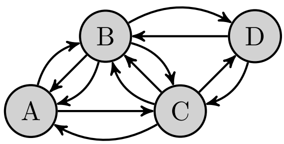
a) Write a matrix representing the number of ways to travel from node to node.
b) How many ways are there of travelling from A to C in a single step?
c) Use the matrix to determine how many ways there are of travelling from A and ending at C in exactly two steps.
d) Use the matrix to determine how many ways there are of travelling from A and ending at C in exactly three steps.
e) Use the matrix to determine how many ways there are of travelling from A and ending at C in less than five steps.

86. Consider the following network:



a) Write a matrix representing the number of ways to travel from node to node.
b) How many ways are there of travelling from A to C in a single step?
c) Use the matrix to determine how many ways there are of travelling from A and ending at C in exactly two steps.
d) Use the matrix to determine how many ways there are of travelling from A and ending at C in exactly three steps.
e) Use the matrix to determine how many ways there are of travelling from A and ending at C in less than five steps.

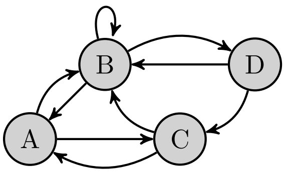
87. Consider the following network:



a) Write a matrix representing the number of ways to travel from node to node.
b) How many ways are there of travelling from A to D in a single step?
c) Use the matrix to determine how many ways there are of travelling from A and ending at D in exactly two steps.

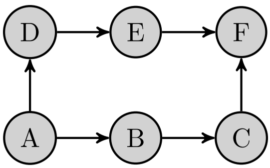
d) Use the matrix to determine how many ways there are of travelling from A and ending at D in exactly three steps.
e) Use the matrix to determine how many ways there are of travelling from A and ending at D in less than six steps.

88. Consider the following network:



a) Write a matrix representing the number of ways to travel from node to node.
b) How many ways are there of travelling from A to D in a single step?
c) Use the matrix to determine how many ways there are of travelling from A and ending at D in exactly two steps.
d) Use the matrix to determine how many ways there are of travelling from A and ending at D in exactly three steps.
e) Use the matrix to determine how many ways there are of travelling from A and ending at D in less than six steps.

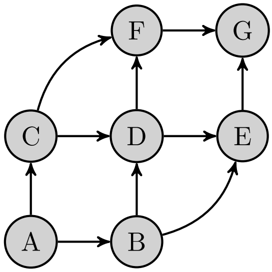
89. Consider the following network:



a) Use a matrix to confirm that there are two ways to travel from A to F in three steps.

b) Show that it is not possible to travel from A to F in fewer than three steps.
c) Show that it is not possible to travel from A to F in more than three steps.
d) Alter the matrix to add a path from B to E. Now how many ways are there to travel from A to F in three steps?

90. Consider the following network:



a) Use a matrix to confirm that there are two ways to travel from A to G in three steps.
b) Show that it is not possible to travel from A to G in fewer than three steps.
c) How many ways is it possible to travel from A to F in more than three steps?
d) Add a path from A to D. How many different ways are there to travel from A to F now?

91. Australian railways run the following services between capitals:

- Perth—Adelaide
- Adelaide—Sydney
- Adelaide—Darwin

- Adelaide—Melbourne
- Sydney—Melbourne
- Sydney—Brisbane

All services run in both directions.

a) Draw a graph to represent the rail network between capitals.

b) Construct an adjacency matrix (i.e. a matrix representing the number of connections between nodes) for the network.

c) Using the adjacency matrix, determine how many ways it is possible to travel starting in Sydney and travelling on three trains before ending in Adelaide.

d) Using the adjacency matrix, determine how many ways it is possible to travel starting in Perth and travelling on five trains before ending back in Perth.

92. An airline runs the following flights in northern Australia:

- Broome—Kununurra
- Broome—Karratha
- Broome—Port Hedland
- Darwin—Kununurra
- Darwin—Broome
- Darwin—Karratha

All services run in both directions.

a) Draw a graph to represent the air transport network.

b) Construct an adjacency matrix (i.e. a matrix representing the number of connections between nodes) for the network.

c) Using the adjacency matrix, determine how many ways it is possible to travel from Kununurra to Karratha in one, two or three legs.

d) The passenger will not want to fly back into Kununurra as one of the stops on a three-leg trip, or out of Karratha once she has arrived there. How would the matrix change to exclude any flights *into* Kununurra or *out* of Karratha? Does this change the calculation of the number of possible ways to travel?

93. Carol meets Glen and Sonya. Glen meets Eddie and Rob. Sonya meets Amy and Eddie. Rob meets Eddie and Amy. In how many different ways can Carol pass a note to Amy if it passes through no more than three people between Carol and Amy? (Your answer should include a graph of the network and the adjacency matrix used, and should take into account that once the note gets to Amy it stops.)

94. Refer to the simplified map in figure 2.1. Ron wants to drive from Albany to Kununurra in such a way that every town or city he visits is further north than the previous one. Represent the road network as a directed graph and then use an adjacency matrix to determine how many different ways he can do this. (Hint: It is not necessary to include every town on the map.)

2.4 Transformations in the Plane

95. Write a matrix for a 90° anticlockwise rotation.

96. Write a matrix for a 90° clockwise rotation.

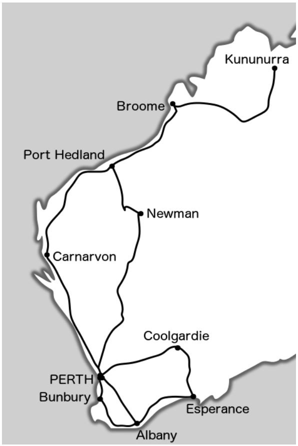


Figure 2.1: Simplified map of Western Australian highways (based on "WAHighways". Licensed under CC BY-SA 3.0 via Wikimedia Commons—https://commons.wikimedia.org/wiki/File:WAHighways.png#/media/File:WAHighways.png)

97. Write a matrix for a 45° clockwise rotation.

98. Write a matrix for a 60° clockwise rotation.

99. Write a matrix for a 120° anticlockwise rotation.

100. Write a matrix for a 150° anticlockwise rotation.

101. Write a matrix for a $-\frac{\pi}{6}$ rotation.

102. Write a matrix for a $\frac{3\pi}{4}$ rotation.

103. Write a matrix for a $-\frac{2\pi}{3}$ rotation.

104. Write a matrix for a $-\frac{3\pi}{2}$ rotation.

105. Write a matrix for a $\frac{5\pi}{4}$ rotation.

106. Write a matrix for a $\frac{3\pi}{4}$ rotation.

107. Write a matrix for a rotation of 95° clockwise.

108. Write a matrix for a rotation of 129° anticlockwise.

109. Write a matrix for a rotation of 72° anticlockwise.

110. Write a matrix for a rotation of 129° anticlockwise.

111. Write a matrix for a rotation of 151° anticlockwise.

112. Write a matrix for a rotation of 38° anticlockwise.

113. Matrix $R = \left[\begin{smallmatrix} 0.1 & a \\ b & c \end{smallmatrix}\right]$ is a rotation matrix. Determine a, b and c.

114. Matrix $R = \begin{bmatrix} a & b \\ \frac{1}{4} & c \end{bmatrix}$ is a rotation matrix. Determine a, b and c.

115. Matrix $R = \begin{bmatrix} a & \frac{4}{5} \\ b & c \end{bmatrix}$ is a rotation matrix resulting in a rotation of less than 90°. Determine a, b and c.

116. Matrix $R = \begin{bmatrix} a & b \\ c & -\frac{5}{12} \end{bmatrix}$ is a rotation matrix resulting in an anticlockwise rotation of less than 180°. Determine a, b and c.

117. Write a matrix for a reflection in the $x-$axis.

118. Write a matrix for a reflection in the $y-$axis.

119. Write a matrix for a reflection in a line that makes a 30° angle with the positive $x-$axis, measured anticlockwise from the axis.

120. Write a matrix for a reflection in a line that makes a 60° angle with the positive $x-$axis, measured anticlockwise from the axis.

121. Write a matrix for a reflection in a line that makes a 60° angle with the positive $x-$axis, measured clockwise from the axis.

122. Write a matrix for a reflection in a line that makes a 30° angle with the positive $x-$axis, measured clockwise from the axis.

123. Write a matrix for a reflection in the line $y = x$.

124. Write a matrix for a reflection in the line $y = -x$.

125. What angle does the line $y = \sqrt{3}x$ make with the positive $x-$axis?

126. What angle does the line $x = \sqrt{3}y$ make with the positive $x-$axis?

127. Given $\tan\theta = 2$, exactly determine $\sin(2\theta)$ (where $-\frac{\pi}{2} < \theta \leqslant \frac{\pi}{2}$).

128. Given $\tan\theta = 2$, exactly determine $\cos(2\theta)$ (where $-\frac{\pi}{2} < \theta \leqslant \frac{\pi}{2}$).

129. Write a matrix for a reflection in the line $y = 2x$.

130. Write a matrix for a reflection in the line $2y = x$.

131. Write a matrix for a reflection in the line $5y = 2x$.

132. Write a matrix for a reflection in the line $3y = 2x$.

133. Write a matrix for a reflection in the line $y = -\frac{2x}{3}$.

134. Write a matrix for a reflection in the line $y = -\frac{3x}{4}$.

135. Matrix $R = \begin{bmatrix} 0.28 & a \\ b & c \end{bmatrix}$ is a reflection matrix. Determine a, b and c.

136. Matrix $R = \begin{bmatrix} a & b \\ c & \frac{9}{10} \end{bmatrix}$ is a reflection matrix. Determine a, b and c.

137. Matrix $R = \begin{bmatrix} a & b \\ \frac{99}{101} & c \end{bmatrix}$ is a reflection matrix where the line of symmetry has gradient $-1 < m < 1$. Determine a, b and c.

138. Matrix $R = \begin{bmatrix} a & b \\ c & -\frac{9}{41} \end{bmatrix}$ is a reflection matrix where the line of symmetry has positive gradient. Determine a, b and c.

139. Explain why the general form for a rotation matrix

$$T = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

gives the same transformation matrix for $\theta = \frac{5\pi}{6}$ as it does for $\theta = -\frac{7\pi}{6}$.

140. Explain why the general form for a reflection matrix

$$T = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

gives the same transformation matrix for $\theta = \frac{3\pi}{8}$ as it does for $\theta = -\frac{5\pi}{8}$.

141. Write a matrix for a uniform dilation by scale factor 5.

142. Write a matrix for a uniform dilation by scale factor 12.

143. Write a matrix for a uniform dilation by scale factor 0.2.

144. Write a matrix for a uniform dilation by scale factor 0.01.

145. Write a matrix for a horizontal dilation by scale factor 3.

146. Write a matrix for a horizontal dilation by scale factor 8.

147. Write a matrix for a vertical dilation by scale factor 3.

148. Write a matrix for a vertical dilation by scale factor 7.

149. Write a matrix for a horizontal dilation by scale factor 4 and vertical dilation by scale factor $\frac{1}{2}$.

150. Write a matrix for a horizontal dilation by scale factor $\frac{1}{3}$ and vertical dilation by scale factor 7.

151. Write a matrix for a horizontal dilation by scale factor 10 and vertical dilation by scale factor 8.

152. Write a matrix for a horizontal dilation by scale factor $\frac{1}{9}$ and vertical dilation by scale factor $\frac{1}{8}$.

153. Describe the dilation achieved by $T = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$

154. Describe the dilation achieved by $T = \begin{bmatrix} 66 & 0 \\ 0 & 66 \end{bmatrix}$

155. Describe the dilation achieved by $T = \begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$

156. Describe the dilation achieved by $T = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{12} \end{bmatrix}$

157. Write a matrix for a shear parallel to the $x-$axis and having shear scale factor of 0.2.

158. Write a matrix for a shear parallel to the $x-$axis and having shear scale factor of 2.1.

159. Write a matrix for a shear parallel to the $x-$axis and having shear scale factor of -2.

160. Write a matrix for a shear parallel to the $x-$axis and having shear scale factor of -1.

161. Write a matrix for a shear parallel to the $y-$axis and having shear scale factor of 1.

162. Write a matrix for a shear parallel to the $y-$axis and having shear scale factor of -1.

163. Write a matrix for a shear parallel to the $y-$axis and having shear scale factor of -0.5.

164. Write a matrix for a shear parallel to the $y-$axis and having shear scale factor of 1.8.

165. Describe the shear transform achieved by $T = \left[\begin{smallmatrix} 1 & 0.01 \\ 0 & 1 \end{smallmatrix}\right]$.

166. Describe the shear transform achieved by $T = \left[\begin{smallmatrix} 1 & 1.5 \\ 0 & 1 \end{smallmatrix}\right]$.

167. Describe the shear transform achieved by $T = \left[\begin{smallmatrix} 1 & 0 \\ -3.1 & 1 \end{smallmatrix}\right]$.

168. Describe the shear transform achieved by $T = \left[\begin{smallmatrix} 1 & -8.2 \\ 0 & 1 \end{smallmatrix}\right]$.

169. Identify $T = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ as rotation, reflection, dilation, shear or other.

170. Identify $T = \left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$ as rotation, reflection, dilation, shear or other.

171. Identify $T = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ as rotation, reflection, dilation, shear or other.

172. Identify $T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$ as rotation, reflection, dilation, shear or other.

173. Identify $T = \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$ as rotation, reflection, dilation, shear or other.

174. Identify $T = \left[\begin{smallmatrix} 2 & 0.5 \\ 0.5 & 2 \end{smallmatrix}\right]$ as rotation, reflection, dilation, shear or other.

175. Write a matrix for a 45° anticlockwise rotation followed by a uniform dilation with scale factor $\sqrt{2}$.

176. Write a matrix for a reflection in the line $y = x$ followed by a uniform dilation with scale factor $\frac{1}{2}$.

177. Write a matrix for a shear parallel to the $x-$axis with shear scale factor 1, followed by a 90° clockwise rotation.

178. Write a matrix for a reflection in the line $5y = x$ followed by a 30° anticlockwise rotation.

179. Write a matrix for a 45° anticlockwise rotation followed by a shear parallel to the $y-$axis with shear scale factor of 1, followed by a 45° clockwise rotation.

180. Write a matrix for a reflection in a line that makes a 15° angle measured anticlockwise from the positive $x-$axis, followed by a shear parallel to the $x - axis$ with shear scale factor 1, followed by another reflection in the same line.

181. Show that a reflection in the $y-$axis followed by a rotation of 90° anticlockwise is equivalent to a reflection in the line $y = -x$.

182. Show that a shear with scale factor 3 parallel to the $x-$axis followed by a rotation of 90° anticlockwise is equivalent to a rotation of 90° anticlockwise followed by a shear with scale factor -3 parallel to the $y-$axis.

183. Show that a shear with scale factor $\frac{4}{3}$ parallel to the $x-$axis followed by a dilation with horizontal scale factor 3 and vertical scale factor 2 is equivalent to the same dilation followed by a shear with scale factor 2 parallel to the $x-$axis.

184. Show that a reflection in the line $y = -x$ followed by a shear parallel to the $y-$axis with shear scale factor 0.5 is equivalent to a shear parallel to the $x-$axis with shear scale factor 0.5 followed by the same reflection.

In the following questions $OABC$ is the rectangle with vertices $O(0,0), A(2,0), B(2,1), C(0,1)$.

185. Determine the area of the figure $O'A'B'C'$ formed by transforming $OABC$ using the matrix $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

186. Determine the area of the figure $O'A'B'C'$ formed by transforming $OABC$ using the matrix $T = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}$.

187. Determine the area of the figure $O'A'B'C'$ formed by transforming $OABC$ using the matrix $T = \begin{bmatrix} -2 & 10 \\ 2 & -10 \end{bmatrix}$.

188. Determine the area of the figure $O'A'B'C'$ formed by transforming $OABC$ using the matrix $T = \begin{bmatrix} -4 & 3 \\ -4 & 6 \end{bmatrix}$.

189. Determine the area of the figure $O'A'B'C'$ formed by transforming $OABC$ using the matrix $T = \begin{bmatrix} -2 & 7 \\ 5 & 5 \end{bmatrix}$.

190. Determine the area of the figure $O'A'B'C'$ formed by transforming $OABC$ using the matrix $T = \begin{bmatrix} -9 & -7 \\ -1 & 7 \end{bmatrix}$.

191. Determine the area of the figure $O'A'B'C'$ formed by transforming $OABC$ using the matrix $T = \begin{bmatrix} -0.8 & -2.4 \\ 2.8 & 0 \end{bmatrix}$.

192. Determine the area of the figure $O'A'B'C'$ formed by transforming $OABC$ using the matrix $T = \begin{bmatrix} 1.5 & 1.5 \\ -0.5 & -1.2 \end{bmatrix}$.

193. What matrix transforms $OABC$ into a square with area= 8units^2?

194. What matrix transforms $OABC$ into a parallelogram with acute angles of $45°$ and area= 8units^2?

195. What matrix transforms $OABC$ so that every point of $O'A'B'C'$ lies on the line $y = 2x$?

196. What matrix transforms $OABC$ so that $A' = C, B' = B$ and $C' = A$?

For the following, plot rectangle $OABC$ before and after transformation by the given matrix and hence describe the transformation in geometric terms. (Label points carefully so as not to be confused by the symmetry of $OABC$.)

197. $T = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

198. $T = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix}$

199. $T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

200. $T = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

201. $T = \begin{bmatrix} 0.5 & 1 \\ -1 & 0 \end{bmatrix}$

202. $T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

203. $T = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

204. $T = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$

205. T is a compound transformation that involves a 45° anti-clockwise rotation followed by a uniform dilation with scale factor $\frac{\sqrt{2}}{2}$ followed by a reflection in the $y-$axis.

a) Determine matrix T.

b) Determine the coordinates of A', B' and C' when points $A(2,0), B(2,1), C(0,1)$ are transformed by T.

c) Determine the area of $OA'B'C'$ (where O is $(0,0)$, i.e. the origin.)

d) Determine matrix V such that V transforms points A', B' and C' back to the original A, B and C.

206. T is a compound transformation that involves a shear parallel to the $y-$axis with shear scale factor 5 followed by a dilation with horizontal scale factor 2 and vertical scale factor 0.2 followed by a rotation of 15° clockwise.

a) Determine matrix T.

b) Determine the coordinates of A', B' and C' when points $A(2,0), B(2,1), C(0,1)$ are transformed by T.

c) Determine the area of $OA'B'C'$ (where O is $(0,0)$, i.e. the origin.)

d) Determine matrix V such that V transforms points A', B' and C' back to the original A, B and C.

2.5 Modelling Dynamic Systems

207. A refining system has three stages: stage 1 represents unrefined stock; stage 2 represents stock in the process of being refined; stage 3 represents refined stock before shipment. The amount of unrefined stock remains constant (i.e. the same amount is received in as is being processed). Every week 75% of the unrefined stock enters stage 2; 30% of the stock in stage 2 moves to stage 3 while 30% stays in stage 2 and 40% is lost as waste; and 20% of stock in stage 3 is sold and leaves the system while the rest remains in stage 3.

a) Write a transition matrix to represent the system.
b) When the system begins operation there are 100 tonnes of raw materials in stage 1. Represent this as a state matrix.
c) How much stock is in stage 3 five weeks after the system begins operation?
d) Eventually the system reaches equilibrium where the refined product going out matches the raw material coming in. At that point, how many tonnes of stock are in the whole system?

208. Fruit is admitted at a canning plant each day during the fruit picking season so that the input stockpile contains 160 tonnes of fruit waiting to be processed. During the day, half of the input stockpile enters the factory and is processed while a further five percent is discarded as damaged. 92% of the daily production run enters theshipping stockpile, 1% is lost as cans are damaged, and the remainder stays in the factory to finish processing on the next day. Each day, three quarters of the shipping stockpile is sent to wholesalers.

a) Write a transition matrix to represent the system.

b) Represent the first day of the picking season (i.e. when the first 160 tonnes arrives) as a state matrix.

c) How much stock is in the shipping stockpile twelve weeks after the system begins operation?

d) If the picking season is long enough, the system will reach equilibrium where the refined product going out matches the raw material coming in. How close is the system to equilibrium at the end of the twelve week picking season?

209. A particular insect's life cycle passes through the phases of egg, larva, pupa and imago. A population is modelled using the following monthly stats (counting only females):

- 10% of eggs hatch into larvae. 20% remain as viable eggs. 70% are eaten by predators.
- 5% of larvae turn into pupae. 20% remain as larvae. 75% are eaten.
- 45% of pupae mature into adults. 50% remain as pupae. 5% die.
- 20% of adults survive for a second month. 80% die. Each adult produces on average 120 viable eggs.

A population of 100 adults is introduced to a new environment. How many adults would you expect to be present after a year? How long would you expect it to take before the adult population returned to 100?

210. The life cycle of a species of slow-growing tree can be modelled in 20-year intervals as follows:

- 0-19 years: 0.01% of seeds germinate and survive to immature trees. Trees do not reproduce at this age.
- 20-39 years: 70% of trees survive this stage. These young trees produce 10 000 seeds each during this stage.
- 40-59 years: 80% of trees survive this stage. Each produces 20 000 seeds. At the end of this stage they become mature trees.
- 60-79 years: 75% of trees survive this stage. Each mature tree produces 30 000 seeds.

Subsequent stages are the same as the 60-79 year (mature tree) stage, so the matrix should be set up so that surviving trees in this stage remain in this stage after each iteration.

A forest rehabilitation area is planted with 250 000 seeds.

a) How many mature trees should we expect to find there after 200 years?
b) One hundred years after the planting, a forestry business is given a one-off licence to harvest all trees that are at least 40 years old. How many trees are harvested?
c) Under the scenario in (b), how many mature trees are there after 200 years?

Solutions

1. Matrix Basics

1. $$\begin{array}{c} \\ N \\ P \\ K \end{array} \begin{array}{c} \begin{array}{cccc} BG & PM & BM & AS \end{array} \\ \begin{bmatrix} 16 & 3 & 4 & 21 \\ 12 & 2 & 12 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

3. $$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 5 & 12 & 0 \\ 5 & 0 & 15 & 3 \\ 12 & 15 & 0 & 2 \\ 0 & 3 & 2 & 0 \end{bmatrix} \end{array}$$

5.
$$\begin{array}{c} \\ \text{from J} \\ \text{from K} \\ \text{from L} \\ \text{from M} \\ \text{from N} \\ \text{from O} \end{array} \begin{array}{c} \begin{array}{cccccc} \text{to J} & \text{to K} & \text{to L} & \text{to M} & \text{to N} & \text{to O} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

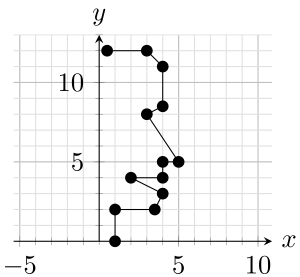
7.
$$\begin{array}{c} \begin{array}{cccccc} 0yo & 1yo & 2yo & 3yo & 4yo & 5yo \end{array} \\ \begin{bmatrix} 25 & 21 & 19 & 28 & 15 & 5 \end{bmatrix} \end{array}$$

9. $\begin{array}{c} M \\ F \end{array} \begin{bmatrix} 8 \\ 7 \end{bmatrix}$

11. (a) 39; (b) $43 + 39 + 40 = 122$; (c) $\frac{36}{231} \approx 0.156$ (3 d.p.); (d) $\frac{227}{685} \approx 0.331$ (3 d.p.)

13. $\begin{bmatrix} 0 & 4 & 2 & 3 \\ 0 & 0 & 4 & 1 \end{bmatrix}$

15.



17. 1×2

19. 2×2

21. 5×1

23. 2×8

25. a) 4; b) no such element; c) -9; d) 2; e) no such element; f) -7; g) 0; h) 9

27. $A = \begin{bmatrix} 11 & 21 & 31 \\ 12 & 22 & 32 \\ 13 & 23 & 33 \end{bmatrix}$

29. $C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \\ 5 & 10 & 15 \end{bmatrix}$

31. $E = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 17 & 18 & 19 & 20 \end{bmatrix}$

33. square, zero.

35. square, identity, diagonal.

37. row.

39. column, zero.

41. (upper) triangular, square.

43. square, column, row, identity (and, possibly, triangular).

45. a) $\begin{bmatrix} 1 & 0 & 5 & 1 \\ 2 & -6 & -2 & 0 \\ 0 & -2 & 8 & 5 \end{bmatrix}$; b) $\begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & -6 & -12 & -2 \\ 0 & -2 & 8 & 5 \end{bmatrix}$; c) $\begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & -6 & -12 & -2 \\ 0 & 0 & 36 & 17 \end{bmatrix}$;

47. a) $\begin{bmatrix} 2 & -8 & 3 & 1 & -1 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 1 & 5 & 10 \\ 8 & -33 & 13 & 16 & 22 \end{bmatrix}$; b) $\begin{bmatrix} 2 & -8 & 3 & 1 & -1 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 1 & 5 & 10 \\ 0 & -1 & 1 & 12 & 26 \end{bmatrix}$; c) $\begin{bmatrix} 2 & -8 & 3 & 1 & -1 \\ 0 & -1 & 1 & 12 & 26 \\ 0 & 0 & 1 & 5 & 10 \\ 0 & 0 & 0 & 4 & -4 \end{bmatrix}$;

49. a) $\begin{bmatrix} 1 & -5 & -7 & 1 & 12 \\ 0 & -15 & -28 & 7 & 86 \\ -2 & -2 & -1 & -7 & -59 \\ -8 & 5 & -6 & 2 & 47 \end{bmatrix}$;

b) $\begin{bmatrix} 1 & -5 & -7 & 1 & 12 \\ 0 & -15 & -28 & 7 & 86 \\ 0 & -12 & -15 & -5 & -35 \\ -8 & 5 & -6 & 2 & 47 \end{bmatrix}$;

c) $\begin{bmatrix} 1 & -5 & -7 & 1 & 12 \\ 0 & -15 & -28 & 7 & 86 \\ 0 & -12 & -15 & -5 & -35 \\ 0 & -35 & -62 & 10 & 143 \end{bmatrix}$;

d) $\begin{bmatrix} 1 & -5 & -7 & 1 & 12 \\ 0 & -15 & -28 & 7 & 86 \\ 0 & 0 & 37 & -53 & -519 \\ 0 & -35 & -62 & 10 & 143 \end{bmatrix}$;

e) $\begin{bmatrix} 1 & -5 & -7 & 1 & 12 \\ 0 & -15 & -28 & 7 & 86 \\ 0 & 0 & 37 & -53 & -519 \\ 0 & 0 & 10 & -19 & -173 \end{bmatrix}$;

f) $\begin{bmatrix} 1 & -5 & -7 & 1 & 12 \\ 0 & -15 & -28 & 7 & 86 \\ 0 & 0 & 37 & -53 & -519 \\ 0 & 0 & 0 & -173 & -1211 \end{bmatrix}$

51. $\begin{bmatrix} 1 & -4 & 1 \\ 0 & 26 & -4 \end{bmatrix}$

53. $\begin{bmatrix} 7 & 3 & -6 \\ 0 & 10 & 10 \end{bmatrix}$

55. $\begin{bmatrix} 3 & 9 & 7 & 0 \\ 0 & 3 & -5 & -5 \\ 0 & 0 & 19 & 16 \end{bmatrix}$

57. $\begin{bmatrix} 1 & 1 & -6 & -2 \\ 0 & 10 & -29 & -18 \\ 0 & 0 & 19 & 10 \end{bmatrix}$

59. $\begin{bmatrix} -8 & -7 & 4 & 10 \\ 0 & -6 & 7 & 17 \\ 0 & 0 & 50 & 31 \end{bmatrix}$

61. $\begin{bmatrix} 1 & 2 & 1 & 2 & 9 \\ 0 & 3 & -5 & 7 & 44 \\ 0 & 0 & 65 & -43 & -203 \\ 0 & 0 & 0 & -2 & -17 \end{bmatrix}$

63. $\begin{bmatrix} 2 & -2 & 1 & -8 & 1 & 0 \\ 0 & -5 & 4 & 0 & 10 & -7 \\ 0 & 0 & 4 & -8 & 7 & 6 \\ 0 & 0 & 0 & -7 & -2 & -5 \\ 0 & 0 & 0 & 0 & -8 & -2 \end{bmatrix}$

65. $\begin{bmatrix} 1 & -0.35 & -1.47 & 0.22 \\ 0 & 1 & 1.47 & 1.02 \\ 0 & 0 & 1 & -0.02 \end{bmatrix}$

67. $\begin{bmatrix} 1 & -0.78 & -0.44 & 0.76 & -3.54 \\ 0 & 1 & 2.77 & 0.23 & -3.46 \\ 0 & 0 & 1 & -0.77 & -16.13 \\ 0 & 0 & 0 & 1 & 17.57 \end{bmatrix}$

69. Only $A+A$, $B+B$, $C+C$, $D+D$ and $E+E$ are possible.

71. $\begin{bmatrix} -1 & -3 \\ -2 & -3 \end{bmatrix}$

73. $\begin{bmatrix} 8 & 0 & -9 \\ -6 & -8 & -5 \end{bmatrix}$

75. $\begin{bmatrix} -12 & 4 & -9 & -5 \end{bmatrix}$

77. The sum is not possible as matrices have different sizes.

79. $\begin{bmatrix} -10 & -6 & -2 & 7 & 6 \\ 13 & -1 & -8 & 2 & 11 \\ -2 & -18 & 16 & -14 & -2 \end{bmatrix}$

81. $\begin{bmatrix} 16 & 1 & 2 & 7 \\ -3 & -7 & 12 & -7 \end{bmatrix}$

83. $\begin{bmatrix} -10 & 7 \\ -3 & -10 \end{bmatrix}$

85. $\begin{bmatrix} -7.8 & -8.5 \end{bmatrix}$

87. The operation is not possible.

89. $\begin{bmatrix} -5.8 & 4.6 & -9.4 & 11.5 \end{bmatrix}$

91. $a = -2.1$; $b = 1.1$; $c = -2.2$

93. $a = 2.4$; $b = 2.0$; $c = 16.5$

95. $A = \begin{bmatrix} 1.5 & -1.3 & -5.7 \\ -8.2 & -8.6 & 4.7 \\ 0.2 & -5.1 & 0.6 \end{bmatrix}$

97. $C = \begin{bmatrix} -2 & 1 & 0 \\ 2 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

99. $\begin{bmatrix} 5 & 20 \\ 20 & -15 \\ 15 & -25 \\ 25 & -25 \end{bmatrix}$

101. $\begin{bmatrix} -4 & -8 & 16 \\ 16 & 12 & 16 \\ -8 & 16 & 16 \\ -20 & 20 & -8 \end{bmatrix}$

103. $\begin{bmatrix} 0 & -3 & 5 \\ 0 & 4 & 2 \\ 5 & 3 & -1 \end{bmatrix}$

105. $\begin{bmatrix} 10 & 0 & -14 \end{bmatrix}$

107. $\begin{bmatrix} -20 & 14 \\ -10 & -12 \\ -13 & 3 \end{bmatrix}$

109. $a = 5$; $b = -3$; $c = -2$; $d = -1$; $e = 0$; $f = 0$

111. $a = -3$; $b = 1$; $c = 5$; $d = -2$

113. Possible: product is (6×4).

115. Possible: product is (4×1).

117. Possible: product is (2×6).

119. Possible: product is (6×6).

121. Possible: product is (1×4).

123. Impossible.

125. $AB = \begin{bmatrix} 6 & -14 & 14 \\ 17 & 20 & 30 \\ 3 & -24 & 13 \end{bmatrix}$ $\quad BA = \begin{bmatrix} 23 & 29 & -22 \\ -25 & 29 & -17 \\ 16 & 18 & -13 \end{bmatrix}$

127. AB is impossible. $\quad BA = \begin{bmatrix} -8 \\ -16 \\ -16 \end{bmatrix}$

129. $AB = \begin{bmatrix} -12 & 6 \end{bmatrix}$ $\quad BA$ is impossible.

131. $AB = \begin{bmatrix} 7 & -8 \\ -18 & 24 \end{bmatrix}$ $\quad BA = \begin{bmatrix} 5 & -13 & 4 & -3 \\ -10 & 26 & -8 & 6 \\ -12 & 15 & -6 & 3 \\ 4 & 22 & -4 & 6 \end{bmatrix}$

133.
$$\begin{aligned}
\text{LHS} &= AB + A \\
&= \begin{bmatrix} 1 & -1 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -4 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3 \\ 1 & -22 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -4 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 2 \\ -3 & -23 \end{bmatrix} \\
\text{RHS} &= A(B + I) \\
&= \begin{bmatrix} 1 & -1 \\ -4 & -1 \end{bmatrix}\left(\begin{bmatrix} 0 & 5 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 1 & -1 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ -1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 2 \\ -3 & -23 \end{bmatrix} \\
&= \text{LHS as required.}
\end{aligned}$$

135.
$$\begin{aligned}
\text{LHS} &= EF + 3E \\
&= \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 4 & -3 \end{bmatrix} + 3\begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 \\ -12 & 10 \end{bmatrix} + \begin{bmatrix} -9 & -6 \\ -3 & -12 \end{bmatrix} \\
&= \begin{bmatrix} -5 & -6 \\ -15 & -2 \end{bmatrix} \\
\text{RHS} &= E(F + 3I) \\
&= \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}\left(\begin{bmatrix} -4 & 2 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) \\
&= \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 4 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -5 & -6 \\ -15 & -2 \end{bmatrix} \\
&= \text{LHS as required.}
\end{aligned}$$

137.
$$\begin{aligned}
\text{LHS} &= (JK)L \\
&= \left(\begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}\right)\begin{bmatrix} -1 & -4 \\ 3 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 5 & -13 \end{bmatrix}\begin{bmatrix} -1 & -4 \\ 3 & -2 \end{bmatrix} \\
&= \begin{bmatrix} -2 & -8 \\ -44 & 6 \end{bmatrix} \\
\text{RHS} &= J(KL) \\
&= \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -1 & -4 \\ 3 & -2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -7 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -2 & -8 \\ -44 & 6 \end{bmatrix} \\
&= \text{LHS as required.}
\end{aligned}$$

139. $\begin{bmatrix} 0.31 & 0.31 \\ 0.69 & 0.69 \end{bmatrix}$

141. $\begin{bmatrix} 4.19 & 3.08 & 4.61 \\ 1.52 & 1.25 & 1.84 \\ 4.80 & 3.83 & 5.66 \end{bmatrix}$

143.
$$\begin{aligned}
\text{LHS} &= (Q+I)^2 \\
&= \left(\begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^2 \\
&= \begin{bmatrix} 2 & 0 \\ -2 & -1 \end{bmatrix}^2 \\
&= \begin{bmatrix} 4 & 0 \\ -2 & 1 \end{bmatrix} \\
\text{RHS} &= Q^2 + 2Q + I \\
&= \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}^2 + 2\begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -4 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 \\ -2 & 1 \end{bmatrix} \\
&= \text{LHS as required.}
\end{aligned}$$

145. 6

147. -43

149. -22

151. 24

153.
$$\begin{aligned}
\text{LHS} &= \det(AB)\\
&= \det\left(\begin{bmatrix} 6 & -3\\ 4 & -3\end{bmatrix}\begin{bmatrix} 1 & -6\\ 4 & 2\end{bmatrix}\right)\\
&= \det\left(\begin{bmatrix} -6 & -42\\ -8 & -30\end{bmatrix}\right)\\
&= 180 - 338 = -156\\
\text{RHS} &= \det(A)\det(B)\\
&= \det\left(\begin{bmatrix} 6 & -3\\ 4 & -3\end{bmatrix}\right)\det\left(\begin{bmatrix} 1 & -6\\ 4 & 2\end{bmatrix}\right)\\
&= (-18+12)(2+24)\\
&= -6\times 26 = -156\\
&= \text{LHS as required.}
\end{aligned}$$

155.
$$\begin{aligned}
\text{LHS} &= \det(3E)\\
&= \det\left(3\begin{bmatrix} 1 & -4\\ 6 & 3\end{bmatrix}\right)\\
&= \det\left(\begin{bmatrix} 3 & -12\\ 18 & 9\end{bmatrix}\right)\\
&= 27 + 216 = 243\\
\text{RHS} &= 3^2\det(E)\\
&= 9\det\left(\begin{bmatrix} 1 & -4\\ 6 & 3\end{bmatrix}\right)\\
&= 9(3+24)\\
&= 9\times 27 = 243\\
&= \text{LHS as required.}
\end{aligned}$$

157. 22

159. 30

161. $a = -5$

163. $c = 7$

165. $e = -4$

167. $g = \pm 8$

169. $k = -3$ or $k = 10$

171. B and D are singular.

173. $-\frac{1}{39}\begin{bmatrix} -3 & 9 \\ 2 & 7 \end{bmatrix}$

175. Matrix is singular; no inverse exists.

177. $-\frac{1}{10}\begin{bmatrix} 4 & 6 \\ -1 & -4 \end{bmatrix}$

179. $-\frac{1}{68}\begin{bmatrix} -4 & 4 \\ 8 & 9 \end{bmatrix}$

181. $\frac{1}{8}\begin{bmatrix} 0 & 2 \\ -4 & -7 \end{bmatrix}$

183.

a) $\left[\begin{array}{ccc:ccc} 1 & -3 & 9 & 1 & 0 & 0 \\ 1 & -7 & -1 & 0 & 1 & 0 \\ 1 & -7 & -4 & 0 & 0 & 1 \end{array}\right]$

b) $$\left[\begin{array}{ccc:ccc} 1 & -3 & 9 & 1 & 0 & 0 \\ 0 & -4 & -10 & -1 & 1 & 0 \\ 0 & -4 & -13 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:ccc} 1 & -3 & 9 & 1 & 0 & 0 \\ 0 & -4 & -10 & -1 & 1 & 0 \\ 0 & 0 & -3 & 0 & -1 & 1 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{ccc:ccc} 1 & -3 & 9 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & 1/4 & -1/4 & 0 \\ 0 & 0 & 1 & 0 & 1/3 & -1/3 \end{array}\right]$$

c) $$\left[\begin{array}{ccc:ccc} 1 & -3 & 9 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/4 & -13/12 & 5/6 \\ 0 & 0 & 1 & 0 & 1/3 & -1/3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:ccc} 1 & -3 & 0 & 1 & -3 & 3 \\ 0 & 1 & 0 & 1/4 & -13/12 & 5/6 \\ 0 & 0 & 1 & 0 & 1/3 & -1/3 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 7/4 & -75/12 & 11/2 \\ 0 & 1 & 0 & 1/4 & -13/12 & 5/6 \\ 0 & 0 & 1 & 0 & 1/3 & -1/3 \end{array}\right]$$

d) $M^{-1} = \frac{1}{12}\begin{bmatrix} 21 & -75 & 66 \\ 3 & -13 & 10 \\ 0 & 4 & -4 \end{bmatrix}$.

Check: $\frac{1}{12}\begin{bmatrix} 1 & -3 & 9 \\ 1 & -7 & -1 \\ 1 & -7 & -4 \end{bmatrix}\begin{bmatrix} 21 & -75 & 66 \\ 3 & -13 & 10 \\ 0 & 4 & -4 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

185. $\frac{1}{2}\begin{bmatrix} -42 & 18 & -8 \\ -7 & 3 & -1 \\ 3 & -1 & 1 \end{bmatrix}$

187. $\frac{1}{11}\begin{bmatrix} -17 & -16 & 19 \\ 20 & 24 & -23 \\ -11 & -11 & 11 \end{bmatrix}$

189. The matrix has no inverse. (Row reduction results in a row of all zeros, an indicator that the matrix is singular.)

191. $P = \begin{bmatrix} 121 & -44 & -88 \\ -59 & 109 & 4 \\ -42 & -82 & -57 \end{bmatrix}$

193. $R = \begin{bmatrix} 12 & -15 & -21 \\ 17 & -65 & -56 \\ 28 & -70 & -49 \end{bmatrix}$

195. $X = T - 3I = \begin{bmatrix} 3 & 10 \\ 4 & -8 \end{bmatrix}$

197. $X = 2I - V = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 7 & -4 \\ -8 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 8 & -3 \end{bmatrix}$

199. $X = U^{-1}V = -\frac{1}{6}\begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 7 & -4 \\ -8 & 5 \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} -24 & 15 \\ -18 & 12 \end{bmatrix} = \begin{bmatrix} 4 & -5/2 \\ 3 & -2 \end{bmatrix}$

201. $X = (U-3I)T^{-1} = \begin{bmatrix} 1 & -3 \\ -2 & -3 \end{bmatrix}(-\frac{1}{70})\begin{bmatrix} -5 & -10 \\ -4 & 6 \end{bmatrix} = -\frac{1}{70}\begin{bmatrix} 7 & -28 \\ 22 & 2 \end{bmatrix}$

203. $X = 10I(T+U)^{-1} = \frac{5}{32}\begin{bmatrix} 5 & 7 \\ 2 & -10 \end{bmatrix}$

205. $X = (5U - 9I)^{-1}T = -\frac{1}{249}\begin{bmatrix} 6 & -165 \\ 104 & 45 \end{bmatrix}$

207. $X = U^{-1}VU = \begin{bmatrix} 21 & -12 \\ 16 & -9 \end{bmatrix}$

209.

$$\begin{aligned}
X &= U + UX + U^2X \\
IX - UX - U^2X &= U \\
(I - U - U^2)X &= U \\
X &= (I - U - U^2)^{-1}U \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -3 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} 22 & -12 \\ -8 & 6 \end{bmatrix}\right)^{-1} \begin{bmatrix} 4 & -3 \\ -2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -25 & 15 \\ 10 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 4 & -3 \\ -2 & 0 \end{bmatrix} \\
&= \frac{1}{-25} \begin{bmatrix} -5 & -15 \\ -10 & -25 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -2 & 0 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -2 & 0 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} -2 & -3 \\ -2 & -6 \end{bmatrix}
\end{aligned}$$

211. Being its own inverse means $W^2 = I$.

$$\begin{aligned}
W^2 &= \begin{bmatrix} w & 2w-1 \\ 1 & -w \end{bmatrix} \begin{bmatrix} w & 2w-1 \\ 1 & -w \end{bmatrix} \\
&= \begin{bmatrix} w^2+2w-1 & 0 \\ 0 & w^2+2w-1 \end{bmatrix} \\
W^2 &= I \\
\begin{bmatrix} w^2+2w-1 & 0 \\ 0 & w^2+2w-1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\therefore \quad w^2 + 2w - 1 &= 1 \\
(w+1)^2 &= 3 \\
w + 1 &= \pm\sqrt{3} \\
w &= -1 \pm \sqrt{3}
\end{aligned}$$

213.

$$\begin{aligned}
\det\left(B^{-1}\right) &= \frac{1}{\det(B)}\\
\text{so } \det(B) &= \det\left(B^{-1}\right)\\
\text{means } \det(B) &= \frac{1}{\det(B)}\\
(\det(B))^2 &= 1\\
\det(B) &= \pm 1
\end{aligned}$$

$$\begin{aligned}
\det(B) &= (b-1)(5-2b)-(3b)(1)\\
&= 5b-2b^2-5+2b-3b\\
&= -2b^2+4b-5\\
\therefore \quad -2b^2+4b-5 &= \pm 1\\
-2(b^2-2b) &= 5\pm 1\\
-2((b-1)^2-1) &= 5\pm 1\\
-2(b-1)^2+2 &= 5\pm 1\\
-2(b-1)^2 &= 3\pm 1\\
(b-1)^2 &= -\frac{3}{2}\mp\frac{1}{2}\\
&= -2 \quad \text{or} \quad -1
\end{aligned}$$

This has no real number solution, but if we allow complex numbers,

$$\begin{aligned}
b-1 &= \pm 2i \quad \text{or} \quad \pm i\\
b &= 1\pm 2i \quad \text{or} \quad 1\pm i
\end{aligned}$$

2. Matrix Applications

1. $x = 3, y = -2$

3. $x = -7, y = 6$

5. $x = \frac{16}{43}, y = -\frac{114}{43}$

7. $x = 9, y = 1, z = -3$

9. $x = -4, y = -8, z = -9$

11. $x = \frac{20}{7}, y = -\frac{8}{7}, z = -\frac{6}{7}$

13. $w = 1, x = -3, y = -2, z = 2$

15. $w = \frac{34}{205}, x = -\frac{407}{205}, y = \frac{53}{41}, z = -\frac{52}{41}$

17. The system has no solution.

19. $a = -\frac{28}{3}$

21. $a = 9$

23. $a = -6, b = 8$

25. $a = -\frac{18}{5}, b = -\frac{25}{3}$

27. $a = -\frac{4}{3}, b = 153$

29.
$$\begin{aligned} -x + 7y &= -3 \\ 4x + 3y &= 7 \end{aligned}$$

31.
$$\begin{aligned} 6y - 5z &= 3 \\ -5x - 9y - 10z &= 2 \\ -x - 5y + 10z &= 3 \end{aligned}$$

33.
$$\begin{aligned} 6w - 2x - 8y - z &= -5 \\ -4w - 7x - 6y + 7z &= 1 \\ -2w - 2x + 2y - 2z &= 1 \\ -8w - 7x + 10y - 5z &= -8 \end{aligned}$$

35.
$$\begin{aligned} 9a + 9b - 6c - 6d - 7e &= -65 \\ -2a + 3b - c - 5d - 5e &= 49 \\ -10a - 3b - 2c - 6d + 4e &= 77 \\ 6a - 2b + 5c + 7d + 3e &= -48 \\ -6a - 10b + 6c + 8d - 9e &= 97 \end{aligned}$$

37. $\begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$

39. $\begin{bmatrix} -3 & -1 & -9 \\ 3 & -8 & 3 \\ 9 & -7 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$

41. $\begin{bmatrix} -6 & 3 & -2 & 2 \\ -7 & 10 & 0 & 10 \\ -3 & -2 & 2 & 5 \\ -9 & 9 & -3 & -5 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 \\ 9 \\ 4 \\ 7 \end{bmatrix}$

43. $\begin{bmatrix} 0 & 1 & 0 & 9 & 0 \\ 9 & 0 & 5 & 0 & 7 \\ 8 & 10 & 0 & 0 & 0 \\ 10 & 0 & 0 & 3 & 8 \\ 4 & 0 & 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23 \\ 41 \\ 8 \\ -58 \\ 4 \end{bmatrix}$

45.
$$\begin{aligned} \begin{bmatrix} -3 & -2 \\ 9 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -13 \\ 4 \end{bmatrix} \\ \begin{bmatrix} -3 & -2 \\ 9 & -1 \end{bmatrix}^{-1}\begin{bmatrix} -3 & -2 \\ 9 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -3 & -2 \\ 9 & -1 \end{bmatrix}^{-1}\begin{bmatrix} -13 \\ 4 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= 1/15\begin{bmatrix} -1 & 2 \\ -9 & -3 \end{bmatrix}\begin{bmatrix} -13 \\ 4 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= 1/15\begin{bmatrix} 21 \\ 105 \end{bmatrix} \\ x &= 7/5, y = 7 \end{aligned}$$

47. $x = 8, y = -10$

49. $x = 7, y = 2$

51. $x = 1.80, y = 7.65$ (2 d.p.)

53.

$$\begin{bmatrix} 14 & -14 & -21 \\ -12 & 6 & 21 \\ -34 & 10 & 42 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -315 \\ 273 \\ 546 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 9 & -4 \\ -5 & -3 & -1 \\ 2 & 8 & -2 \end{bmatrix}\begin{bmatrix} 14 & -14 & -21 \\ -12 & 6 & 21 \\ -34 & 10 & 42 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 9 & -4 \\ -5 & -3 & -1 \\ 2 & 8 & -2 \end{bmatrix}\begin{bmatrix} -315 \\ 273 \\ 546 \end{bmatrix}$$

$$\begin{bmatrix} 42 & 0 & 0 \\ 0 & 42 & 0 \\ 0 & 0 & 42 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 9 & -4 \\ -5 & -3 & -1 \\ 2 & 8 & -2 \end{bmatrix}\begin{bmatrix} -315 \\ 273 \\ 546 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{1}{42}\begin{bmatrix} -315+9\times 273-4\times 546 \\ 5\times 315-3\times 273-546 \\ -2\times 315+8\times 273-2\times 546 \end{bmatrix} \\ &= \frac{1}{42}\begin{bmatrix} -315+2457-2184 \\ 1575-819-546 \\ -630+2184-1092 \end{bmatrix} \\ &= \frac{1}{42}\begin{bmatrix} -42 \\ 210 \\ 462 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 5 \\ 11 \end{bmatrix} \end{aligned}$$

$$x = -1, y = 5, z = 11$$

55.

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & -1 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 & -16 & -7 \\ -4 & -1 & 2 & 0 \\ 1 & 2 & -4 & 0 \\ -6 & -12 & 17 & 7 \end{bmatrix}\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & -1 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & 8 & -16 & -7 \\ -4 & -1 & 2 & 0 \\ 1 & 2 & -4 & 0 \\ -6 & -12 & 17 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 1 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & 8 & -16 & -7 \\ -4 & -1 & 2 & 0 \\ 1 & 2 & -4 & 0 \\ -6 & -12 & 17 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 1 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} 12+32-16-63 \\ -12-4+2 \\ 3+8-4 \\ -18-48+17+63 \end{bmatrix}$$

$$= -\frac{1}{7}\begin{bmatrix} -35 \\ -14 \\ 7 \\ 14 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 2 \\ -1 \\ -2 \end{bmatrix}$$

$$w = 5, x = 2, y = -1, z = -2$$

57.

$$\begin{aligned} v - x + w &= 5 \\ v + x + 2y - 4z &= 8 \\ -v - w + x + z &= -1 \\ v - w + y &= 7 \\ v - w - x + 3y &= 1 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 2 & -4 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ -1 \\ 7 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 1 & 4 & 7 & -3 \\ 15 & 3 & 12 & -6 & 0 \\ 14 & 4 & 16 & 1 & -3 \\ 7 & 2 & 8 & -4 & 3 \\ 9 & 0 & 9 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 2 & -4 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 & 1 & 4 & 7 & -3 \\ 15 & 3 & 12 & -6 & 0 \\ 14 & 4 & 16 & 1 & -3 \\ 7 & 2 & 8 & -4 & 3 \\ 9 & 0 & 9 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \\ -1 \\ 7 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 40+8-4+49-3 \\ 75+24-12-42 \\ 70+32-16+7-3 \\ 35+16-8-28+3 \\ 45-9 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} &= \frac{1}{9} \begin{bmatrix} 90 \\ 45 \\ 90 \\ 18 \\ 36 \end{bmatrix} \\ &= \begin{bmatrix} 10 \\ 5 \\ 10 \\ 2 \\ 4 \end{bmatrix} \end{aligned}$$

$$v = 10, w = 5, x = 10, y = 2, z = 4$$

59. Let $N = 10x + y$

Assuming $x < y$ we get

$$\begin{aligned}
10x + y &= 13(y - x) + 1 \\
\text{i.e. } 23x - 12y &= 1 \\
10x + y &= 8(x + y) + 4 \\
\text{i.e. } 2x - 7z &= 4 \\
\begin{bmatrix} 23 & -12 \\ 2 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 23 & -12 \\ 2 & -7 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
&= \frac{1}{-137}\begin{bmatrix} -7 & 12 \\ -2 & 23 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
&= \frac{1}{-137}\begin{bmatrix} 41 \\ 90 \end{bmatrix}
\end{aligned}$$

but we need x and y to be single digit positive integers, so we must have $x > y$. This gives a different first equation:

$$\begin{aligned}
10x + y &= 13(x - y) + 1 \\
\text{i.e. } -3x + 14y &= 1 \\
\begin{bmatrix} -3 & 14 \\ 2 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -3 & 14 \\ 2 & -7 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
&= \frac{1}{-7}\begin{bmatrix} -7 & -14 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
&= \frac{1}{-7}\begin{bmatrix} -63 \\ -14 \end{bmatrix} \\
&= \begin{bmatrix} 9 \\ 2 \end{bmatrix} \\
\text{i.e. } x = 9, y &= 2 \\
\therefore N &= 92
\end{aligned}$$

61. $N = 39$

63.

$$\begin{aligned}
\text{Let } N &= 100x + 10y + z \\
100x + 10y + z &= 21(x + y + z) + 11 \\
\text{i.e. } 79x - 11y - 20z &= 11 \\
x + 10y + 100z &= 100x + 10y + z + 495 \\
\text{i.e. } -99x + 99z &= 495 \\
\text{i.e. } -x + z &= 5 \\
y &= 2x \\
\text{i.e. } -2x + y &= 0 \\
\begin{bmatrix} 79 & -11 & -20 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 11 \\ 5 \\ 0 \end{bmatrix} \\
\text{Solving gives } x = 3, y = 6, z &= 8 \\
\therefore N &= 368
\end{aligned}$$

65.

a) Let S be the stock matrix, $S = \begin{bmatrix} 6 & 28 & 27 \\ 18 & 7 & 23 \\ 8 & 18 & 6 \\ 0 & 4 & 9 \end{bmatrix}$. Determine the total in each row (i.e. of each hat type) by post-multiplying by $T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. To get the total values, pre-multiply by a row matrix with the value for each style: $V = \begin{bmatrix} 25 & 18 & 3.50 & 49 \end{bmatrix}$. Thus the total stock value is

$$\begin{aligned}
\text{total value} &= VST \\
&= \begin{bmatrix} 25 & 18 & 3.50 & 49 \end{bmatrix}\begin{bmatrix} 6 & 28 & 27 \\ 18 & 7 & 23 \\ 8 & 18 & 6 \\ 0 & 4 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \$3\,138
\end{aligned}$$

b) To count each large size as 0.7 of its full value (i.e. a 30% discount) the simplest approach is to change the matrix T used to total the different sizes: $T' = \begin{bmatrix} 1 \\ 1 \\ 0.7 \end{bmatrix}$. The total stock value now is

$$\begin{aligned} \text{total value} &= VST' \\ &= \begin{bmatrix} 25 & 18 & 3.50 & 49 \end{bmatrix} \begin{bmatrix} 6 & 28 & 27 \\ 18 & 7 & 23 \\ 8 & 18 & 6 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0.7 \end{bmatrix} \\ &= \$2\,672.70 \end{aligned}$$

c) The marked up price can be achieved with a scalar multiple of 1.5 times the stock value, but we also need to subtract \$1.20 per hat. This gives an effective selling value of $V' = 1.5\begin{bmatrix} 25 & 18 & 3.50 & 49 \end{bmatrix} - \begin{bmatrix} 1.20 & 1.20 & 1.20 & 1.20 \end{bmatrix}$. This gives a total revenue of

$$\begin{aligned} \text{revenue} &= V'ST' \\ &= (1.5\begin{bmatrix} 25 & 18 & 3.50 & 49 \end{bmatrix} - \begin{bmatrix} 1.20 & 1.20 & 1.20 & 1.20 \end{bmatrix}) \\ &\quad \times \begin{bmatrix} 6 & 28 & 27 \\ 18 & 7 & 23 \\ 8 & 18 & 6 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0.7 \end{bmatrix} \\ &= \$3\,847.64 \end{aligned}$$

67.

a) $$\begin{bmatrix} 17 & 2 & 0 & 2022 & 1437 \\ 14 & 5 & 0 & 2194 & 1647 \\ 13 & 4 & 2 & 2162 & 1632 \\ 13 & 5 & 1 & 1781 & 1642 \\ 12 & 7 & 0 & 1760 & 1454 \\ 12 & 7 & 0 & 1891 & 1722 \\ 10 & 8 & 1 & 1591 & 1696 \\ 9 & 10 & 0 & 1659 & 1565 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 19 \\ 19 \\ 19 \\ 19 \\ 19 \\ 19 \\ 19 \\ 19 \end{bmatrix}$$

b) $$\begin{bmatrix} 17 & 2 & 0 & 2022 & 1437 \\ 14 & 5 & 0 & 2194 & 1647 \\ 13 & 4 & 2 & 2162 & 1632 \\ 13 & 5 & 1 & 1781 & 1642 \\ 12 & 7 & 0 & 1760 & 1454 \\ 12 & 7 & 0 & 1891 & 1722 \\ 10 & 8 & 1 & 1591 & 1696 \\ 9 & 10 & 0 & 1659 & 1565 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 68 \\ 56 \\ 56 \\ 54 \\ 48 \\ 48 \\ 42 \\ 36 \end{bmatrix}$$

c) $$\begin{bmatrix} 17 & 2 & 0 & 2022 & 1437 \\ 14 & 5 & 0 & 2194 & 1647 \\ 13 & 4 & 2 & 2162 & 1632 \\ 13 & 5 & 1 & 1781 & 1642 \\ 12 & 7 & 0 & 1760 & 1454 \\ 12 & 7 & 0 & 1891 & 1722 \\ 10 & 8 & 1 & 1591 & 1696 \\ 9 & 10 & 0 & 1659 & 1565 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 685 \\ 547 \\ 530 \\ 139 \\ 306 \\ 169 \\ -105 \\ 94 \end{bmatrix}$$

69. Ignoring the cost initially,

$$\begin{bmatrix} b & p & w \end{bmatrix} \begin{bmatrix} 0.12 & 0.06 & 0.02 \\ 0 & 0.061 & 0 \\ 0 & 0.015 & 0.07 \end{bmatrix} = \begin{bmatrix} 12 & 10 & 6 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} b & p & w \end{bmatrix} &= \begin{bmatrix} 12 & 10 & 6 \end{bmatrix} \begin{bmatrix} 0.12 & 0.06 & 0.02 \\ 0 & 0.061 & 0 \\ 0 & 0.015 & 0.07 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 100 & 51.5 & 57.1 \end{bmatrix} \end{aligned}$$

and now calculating the cost

$$\begin{bmatrix} 100 & 51.5 & 57.1 \end{bmatrix} \begin{bmatrix} 0.15 \\ 0.12 \\ 0.03 \end{bmatrix} = \begin{bmatrix} 22.90 \end{bmatrix}$$

Graham should use 100kg of bird guano, 52kg of phosphate rock and 57kg of wood ash at a total ingredient cost of around \$23. (Note: you should not give your answer specified with more digits of precision than your inputs. Round appropriately.)

71. The mixture is 41% Almond, 31% Brazil and 28% Cashew.

73.

$$
\begin{aligned}
f(x) &= ax^2 + bx + c \\
\begin{bmatrix} (-1)^2 & -1 & 1 \\ 1^2 & 1 & 1 \\ 2^2 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix} \\
\begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} (-1)^2 & -1 & 1 \\ 1^2 & 1 & 1 \\ 2^2 & 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix} \\
\therefore \quad f(x) &= -2x^2 - x + 6
\end{aligned}
$$

75. $f(x) = -\frac{11}{120}x^3 + \frac{23}{20}x^2 - \frac{349}{120}x - \frac{123}{20}$

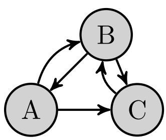
77.

$$
\textit{To:}\quad
\begin{array}{c c}
 & \textit{From:} \\
 & \begin{array}{ccc} A & B & C \end{array} \\
\begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}
\end{array}
$$

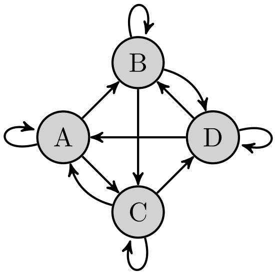
79.

$$
\textit{To:}\quad
\begin{array}{c c}
 & \textit{From:} \\
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}
\end{array}
$$

81.



83.



85.

a)
$$\begin{array}{cc} & \textit{From:} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \textit{To:} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

b) One.

c) $\begin{bmatrix} 1&0&1 \\ 1&1&0 \\ 1&1&1 \end{bmatrix}^2 = \begin{bmatrix} 2&1&2 \\ 2&1&1 \\ 3&2&2 \end{bmatrix}$
There are three ways to go from A to C in two steps.

d) $\begin{bmatrix} 1&0&1 \\ 1&1&0 \\ 1&1&1 \end{bmatrix}^3 = \begin{bmatrix} 5&3&4 \\ 4&2&3 \\ 7&4&5 \end{bmatrix}$
There are seven ways to go from A to C in three steps.

e) $\begin{bmatrix} 1&0&1 \\ 1&1&0 \\ 1&1&1 \end{bmatrix}^4 = \begin{bmatrix} 12&7&9 \\ 9&5&7 \\ 16&9&12 \end{bmatrix}$
There are $1 + 3 + 7 + 16 = 27$ ways to go from A to C in fewer than five steps.

87.

a) $$\text{To:}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \textit{From:} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b) Zero.

c) $\begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 3 & 1 & 4 & 3 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 4 & 1 \\ 2 & 1 & 2 & 2 \end{bmatrix}$

There are two ways to go from A to D in two steps.

d) $\begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 5 & 13 & 8 & 5 \\ 7 & 8 & 14 & 7 \\ 7 & 7 & 8 & 7 \\ 3 & 8 & 6 & 3 \end{bmatrix}$

There are three ways to go from A to D in three steps.

e) $\begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^4 + \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^5 = \begin{bmatrix} 80 & 122 & 124 & 80 \\ 87 & 131 & 144 & 87 \\ 72 & 102 & 116 & 72 \\ 51 & 79 & 80 & 51 \end{bmatrix}$

There are $0 + 2 + 3 + 51 = 56$ ways to go from A to D in fewer than six steps.

89.

a) $$N = \text{To:}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \textit{From:} \\ \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

b) $N^3 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 2&0&0&0&0&0 \end{bmatrix}$

c) $N + N^2 = \begin{bmatrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 1&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 1&0&0&1&0&0 \\ 0&1&1&1&1&0 \end{bmatrix}$

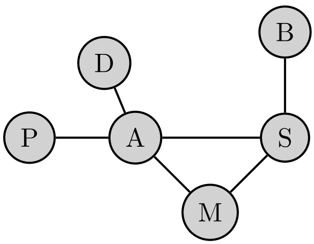
d)

$$N' = \textit{To:} \quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \textit{From:} \\ \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&1&0&0 \\ 0&0&1&0&1&0 \end{bmatrix} \end{array}$$

$(N')^3 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 3&0&0&0&0&0 \end{bmatrix}$

There are now three ways to get from A to F in three moves.

91.

a) 

b)
$$\begin{array}{cc} & \textit{From:} \\ \textit{To:} & \begin{array}{c|cccccc} & P & A & D & M & S & B \\ \hline P & 0 & 1 & 0 & 0 & 0 & 0 \\ A & 1 & 0 & 1 & 1 & 1 & 0 \\ D & 0 & 1 & 0 & 0 & 0 & 0 \\ M & 0 & 1 & 0 & 0 & 1 & 0 \\ S & 0 & 1 & 0 & 1 & 0 & 1 \\ B & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \end{array}$$

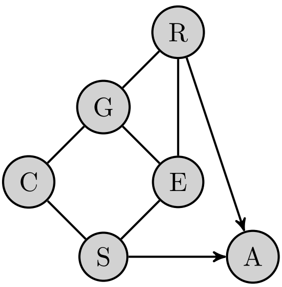
c)
$$\begin{bmatrix} 0&1&0&0&0&0 \\ 1&0&1&1&1&0 \\ 0&1&0&0&0&0 \\ 0&1&0&0&1&0 \\ 0&1&0&1&0&1 \\ 0&0&0&0&1&0 \end{bmatrix}^3 = \begin{bmatrix} 0&4&0&1&1&1 \\ 4&2&4&5&6&1 \\ 0&4&0&1&1&1 \\ 1&5&1&2&4&1 \\ 1&6&1&4&2&3 \\ 1&1&1&1&3&0 \end{bmatrix}$$

There are six ways of starting at Sydney, travelling on three trains, and finishing in Adelaide.

d)
$$\begin{bmatrix} 0&1&0&0&0&0 \\ 1&0&1&1&1&0 \\ 0&1&0&0&0&0 \\ 0&1&0&0&1&0 \\ 0&1&0&1&0&1 \\ 0&0&0&0&1&0 \end{bmatrix}^5 = \begin{bmatrix} 2&19&2&8&8&6 \\ 19&20&19&27&33&8 \\ 2&19&2&8&8&6 \\ 8&27&8&16&21&8 \\ 8&33&8&21&18&13 \\ 6&8&6&8&13&2 \end{bmatrix}$$

There are two ways of starting in Perth, travelling on exactly five trains, and ending back in Perth.

93.



$$N = \textit{To:}\quad \begin{array}{cc} & \textit{From:} \\ & \begin{array}{cccccc} C & G & S & R & E & A \end{array} \\ \begin{array}{c} C \\ G \\ S \\ R \\ E \\ A \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$N^2 + N^3 + N^4 = \begin{bmatrix} 11 & 8 & 5 & 9 & 14 & 0 \\ 8 & 20 & 14 & 13 & 14 & 0 \\ 5 & 14 & 11 & 9 & 8 & 0 \\ 9 & 13 & 9 & 12 & 13 & 0 \\ 14 & 14 & 8 & 13 & 20 & 0 \\ 8 & 9 & 5 & 7 & 12 & 0 \end{bmatrix}$$

The note can get to Amy in eight ways.

95. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

97. $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

99. $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

101. $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

103. $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

105. $\begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

107. $\begin{bmatrix} -0.087 & 0.996 \\ -0.996 & -0.087 \end{bmatrix}$ (3 d.p.)

109. $\begin{bmatrix} 0.309 & -0.951 \\ 0.951 & 0.309 \end{bmatrix}$ (3 d.p.)

111. $\begin{bmatrix} -0.875 & -0.485 \\ 0.485 & -0.875 \end{bmatrix}$ (3 d.p.)

113. $\cos\theta = 0.1$ so $\sin\theta = \pm\sqrt{0.99} = \pm\frac{3\sqrt{11}}{10}$ giving either $a = -\frac{3\sqrt{11}}{10}, b = \frac{3\sqrt{11}}{10}, c = 0.1$ or $a = \frac{3\sqrt{11}}{10}, b = -\frac{3\sqrt{11}}{10}, c = 0.1$.

115. $-\sin\theta = \frac{4}{5}$ so $\cos\theta = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$. A rotation of less than 90° means θ is in the 1st or 4th quadrant (depending on whether the rotation is clockwise or anticlockwise), in either case cosine is positive so $\cos\theta = \frac{3}{5}$ giving $a = \frac{3}{5}, b = -\frac{4}{5}, c = \frac{3}{5}$. (Since $\sin\theta < 0$ this is a clockwise rotation, but it's not necessary to note that in order to answer the question.)

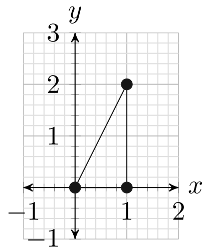
117. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

119. $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

121. $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

123. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

125. $\tan\theta = \sqrt{3} \implies \theta = 60^\circ$ (or $\frac{\pi}{3}$ radians) anticlockwise.

127. 

In the diagram, the hypotenuse (with gradient 2) has length $\sqrt{5}$ so we have $\sin\theta = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$. Using the double angle formula,

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta\\ &= 2\times\frac{2}{\sqrt{5}}\times\frac{1}{\sqrt{5}}\\ &= \frac{4}{5}\end{aligned}$$

129. $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

131. $\begin{bmatrix} \frac{21}{29} & \frac{20}{29} \\ \frac{20}{29} & -\frac{21}{29} \end{bmatrix}$

133. $\begin{bmatrix} \frac{5}{13} & -\frac{12}{13} \\ -\frac{12}{13} & -\frac{5}{13} \end{bmatrix}$

135. $\cos 2\theta = 0.28$ so $\sin 2\theta = \pm\sqrt{0.9216} = \pm 0.96$ giving either $a = b = 0.94, c = -0.28$ or $a = b = -0.96, c = -0.28$.

137. $\sin 2\theta = \frac{99}{101}$ so $\cos 2\theta = \pm\sqrt{1 - \left(\frac{99}{101}\right)^2} = \pm\frac{20}{101}$. A line of symmetry gradient $-1 < m < 1$ means $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ so $\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$ and cosine is positive, giving $\cos 2\theta = \frac{20}{101}$. Hence $a = \frac{20}{101}, b = \frac{99}{101}, c = -\frac{20}{101}$.

139. This can be explained in two ways. Sine and cosine are periodic functions with period 2π. $\theta = \frac{5\pi}{6}$ and $\theta = -\frac{7\pi}{6}$ are separated by 2π so they give the same results for $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Thinking geometrically, a clockwise rotation of $\frac{7\pi}{6}$ arrives at the same point as an anticlockwise rotation of $\frac{5\pi}{6}$.

141. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

143. $\begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$

145. $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

147. $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

149. $\begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

151. $\begin{bmatrix} 10 & 0 \\ 0 & 8 \end{bmatrix}$

153. Uniform dilation by scale factor 1.5.

155. Dilation with horizontal scale factor 4 and vertical scale factor $\frac{1}{4}$.

157. $\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$

159. $\begin{bmatrix} 1 & 12 \\ 0 & 1 \end{bmatrix}$

161. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

163. $\begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix}$

165. Shear parallel to the $x-$axis with shear scale factor of 0.01.

167. Shear parallel to the $y-$axis with shear scale factor of -3.1.

169. Reflection (in line of symmetry with angle $\theta = 45°$)

171. Rotation ($90°$ anticlockwise)

173. Shear (parallel to $y-$axis, shear scale factor 2)

175. $T = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

177. $T = \begin{bmatrix} \sqrt{0} & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$

179. $T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$

181. Reflection in the $y-$axis followed by a rotation of $90°$ anticlockwise:

$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Reflection in the line $y = -x$ (i.e. $\theta = -\frac{\pi}{4}$):

$T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

183. Shear followed by dilation:

$T = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$

Dilation followed by shear:

$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$

185. Area= 2units^2.

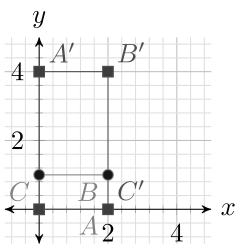
187. Area= 0units^2.

189. Area= 90units^2.

191. Area= 13.44units^2.

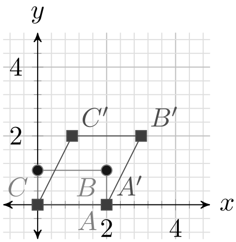
193. The square $O'A'B'C$ must have side length $\sqrt{8} = 2\sqrt{2}$ units. That requires a horizontal dilation of $\sqrt{2}$ and a vertical dilation of $2\sqrt{2}$ resulting in $T = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}$. (Other answers are possible, since the transformation could include a reflection or rotation and still satisfy the requirements.)

195. There are many possible answers. The simplest is something like $T = \left[\begin{smallmatrix} 1 & 0 \\ 2 & 0 \end{smallmatrix}\right]$ but any matrix having the second row equal to double the first row will satisfy the requirements.

197. 

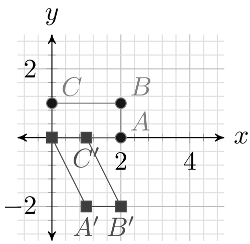
T is a reflection in the line $y = x$ combined with a dilation with scale factor 2 (in either order).

199.



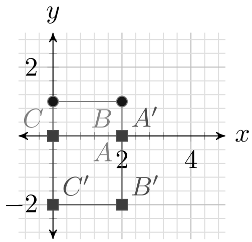
T is a shear parallel to the $x-$axis with shear scale factor 1, followed by a dilation with vertical scale factor 2 and horizontal scale factor 1. Alternatively, T is a dilation with vertical scale factor 2 and horizontal scale factor 1, followed by a shear parallel to the $x-$axis with shear scale factor 0.5.

201.



T is a shear parallel to the $y-$axis with shear scale factor 0.5 followed by a 90° clockwise rotation, or equivalently, a 90° clockwise rotation followed by a shear parallel to the $x-$axis with shear scale factor -0.5.

203.



T is a reflection in the $x-$axis and a dilation with vertical scale factor 2 and horizontal scale factor 1, in either order.

205.

a) Let R be the rotation, D the dilation and F the reflection.

$$R = \begin{bmatrix} \cos\left(45^\circ\right) & -\sin\left(45^\circ\right) \\ sin(45^\circ) & \cos\left(45^\circ\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} T &= FDR \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

b)

$$\begin{aligned}[A'B'C'] &= T[ABC] \\ &= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \end{bmatrix} \\ \text{i.e. } A' &= (-1, 1), \quad B' = (-\tfrac{1}{2}, \tfrac{3}{2}), \quad C' = (\tfrac{1}{2}, \tfrac{1}{2})\end{aligned}$$

c)

$$\begin{aligned}\text{Area } OABC &= 2 \\ \text{Area } OA'B'C' &= 2|\det(T)| \\ &= 2 \times |-\tfrac{1}{4} - \tfrac{1}{4}| \\ &= 1\end{aligned}$$

d)

$$\begin{aligned}V &= T^{-1} \\ &= \frac{1}{-0.5}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\end{aligned}$$

207.

a)
$$T = \begin{array}{c} \\ \textit{To:} \end{array} \begin{array}{c} \\ S1 \\ S2 \\ S2 \end{array} \begin{array}{c} \textit{From:} \\ \begin{array}{ccc} S1 & S2 & S3 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0.75 & 0.3 & 0 \\ 0 & 0.3 & 0.8 \end{array}\right] \end{array}$$

b) $S = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$

c) $T^5 S = \begin{bmatrix} 100 \\ 106.9 \\ 87.1 \end{bmatrix}$: There are 87.1 tonnes in stage 3 five weeks after the system begins operation.

d) 367.8 tonnes.

209. After 12 months we expect to find about 24 adults. It will take about 164 months (13 years 8 months) for there to be 100 adults again.