Topics in Secondary Mathematics

Complex Numbers

Glen Prideaux

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For my students

Contents

Preface

I have often found myself saying to teacher colleagues that what I *really* want from a text book is a set of well designed, graded practice problems for students to work through. I don't need the book to contain explanations and examples; I'll give my students what explanations and examples they need, and if they want more there are numerous places they can go on the Internet to get more. *Topics in Secondary Mathematics* sets out to be such a resource. I intend to include a large number of questions of graded difficulty and complexity with answers to odd numbered questions (so students can get immediate feedback while also allowing teachers to validate students' work) and some fully worked solutions.

The contents of this book are influenced by the Australian curriculum, but no attempt has been made to follow any specific curriculum boundaries. This is a deliberate attempt to help teachers avoid the temptation of teaching to a text book rather than the official curriculum.

Acknowledgements

Thanks go firstly to my wife Carol for her unwavering encouragement and support. Thanks also to colleagues who have provided encouragement. Thanks especially to my students who have used this resource and have helped to identify and correct errors.

1 Complex Numbers as $a + bi$

1.1 Identification

Identify each of the following as real, imaginary or complex. You should assume that x is a real number.

In this section the word 'complex' is used in a non-standard way to mean numbers that have non-zero real and imaginary components.

1. $\sqrt{5}$

2. $\sqrt{-17.5}$

3. $\sqrt[3]{-28}$

4. $\sqrt{-9}$

5. $\sqrt{5-9}$

6. $\sqrt[3]{7.2-18.5}$

7. $\sqrt{9-5}$

8. $\sqrt{9}-\sqrt{5}$

9. $\sqrt{-13}+\sqrt{-52}$

10. $\sqrt{27}-\sqrt{2.7}$

11. $\sqrt{5}-\sqrt{9}$

12. $\sqrt{3}+\sqrt{-1}$

13. $\sqrt{5}+\sqrt{-9}$

14. $\sqrt{-0.2}-\sqrt{-0.01}$

15. $-\sqrt{-5}+\sqrt{9}$

16. $-\sqrt{-9}-\sqrt{-4}$

17. $\sqrt{-5}+\sqrt{-9}$

18. $\sqrt{2\pi-5}$

19. $\sqrt{17-2\pi}$

20. $\sqrt{2\pi-\sqrt{16}}$

21. $(10-5\pi)^{\frac{1}{2}}$

22. $35+2\sqrt{-100}$

23. $\frac{\sqrt{52}}{\sqrt{-13}}$

24. $\frac{\sqrt{-99}}{-\sqrt{-3}}$

25. $\left(x^2\right)^{-\frac{1}{2}}$

26. $(\sqrt{-3.8})^2$

27. $(\sqrt{-5.7})^{-3}$

28. $\sqrt{-2x^2 + 3x - 5}$

29. $\sqrt{x^2 + x + 5}$

30. $(1 - \sqrt{-2})(1 + \sqrt{2})$

31. $(1 - \sqrt{-5})(1 + \sqrt{-5})$

32. $(-1 - x^2)^{\frac{1}{2}}$

Determine what values of x (if any) result in the expression being real, imaginary and complex:

33. $\sqrt{x-1}$

34. $\sqrt{5-x}$

35. $5 + \sqrt{9 - 4x}$

36. $7.3 - \sqrt{0.36x - 6}$

37. $\sqrt{\frac{1}{x} - 1}$

38. $\sqrt{\frac{1}{x-1}}$

39. $x - \sqrt{x}$

40. $2x + \sqrt{x + 2}$

41. $\sqrt{x^2 + 5x - 6}$

42. $x\pi - \sqrt{(x-1)^2 + 2}$

43. $(x-2)^2 + \sqrt{x-5}$

44. $\sqrt{-x^2 + 10x - 25} - x^2 - 3x - 2$

45. $\frac{1+\sqrt{x^2-16}}{2x+10}$

46. $\frac{1}{\sqrt{x-5}\sqrt{3-2x}}$

47. $\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x+1}}$

48. $\frac{\sqrt{x^2-16}}{\sqrt{4-x}}$

1.2 Modulus

Determine the modulus of the following complex numbers. (a is a real number.)

49. $1 + 6i$

50. $10 + 2i$

51. $-4 - 10i$

52. $-7 + 5i$

53. $5 + 5ai$

54. $-5 + 8ai$

55. $-5a + 4a^2 i$

56. $5a^2 - 10ai$

1.3 Complex Conjugates

For each complex number z given below, give its complex conjugate $\bar{z}$. (Assume pronumerals other than z represent real numbers.)

57. $z = 3 + 4i$

58. $z = \sqrt{2} + 8i$

59. $z = -2 - \sqrt{3}i$

60. $z = -1 - 6i$

61. $z = 29i$

62. $z = 3\pi - 5$

63. $z = 1.1i - 3$

64. $z = e^3 + \frac{i}{4}$

65. $z = \frac{5+\sqrt{25-4\times2\times6}}{2\times2}$

66. $z = \frac{-1-\sqrt{1-4\times1\times3}}{2\times1}$

67. $z = \frac{1}{r} + 5ci$

68. $z = \frac{e}{50} - \frac{e}{h}i$

1.4 Complex Arithmetic

Simplify:

69. $(3 - 7i) + (5 + 2i)$

70. $(-3i) + (-7 + 2i)$

71. $(8i) + (8 + 7i)$

72. $(-5 - i) + (6 + 3i)$

73. $(-6.9 + 4.0i) + (-10.6 + 7.4i)$

74. $(-5.8 + 8.0i) + (4.8 + 0.9i)$

75. $(4.4 - 1.3i) + (5.8)$

76. $(2.8 - 3.7i) + (5.5 + 8.9i)$

77. $(-8 + 3i) - (6 + 8i)$

78. $(-5 + 9i) - (3 + 6i)$

79. $(9 + 3i) - (3 - 7i)$

80. $(-3 + 7i) - (10 + 9i)$

81. $(6.0 + 4.4i) - (-7.8 + 6.0i)$

82. $(4 + 7.1i) - (-7.3 - 3.5i)$

83. $(-10.7 - 7.5i) - (-9.9 + 5.0i)$

84. $(-1.7 + 4.7i) - (1.6 - 7.3i)$
85. $(-5)(-2i)$
86. $(2)(-7i)$
87. $(-8i)(3i)$
88. $(-4i)(10i)$
89. $(-5)(-2 - 6i)$
90. $(2)(-3 + 8i)$
91. $(-8i)(4 - i)$
92. $(-3i)(8 - i)$
93. $(1 + 4i)(7 + 7i)$
94. $(1 + 7i)(6 + 3i)$
95. $(7 - 5i)(9 - 10i)$
96. $(9 - 7i)(1 - 4i)$
97. $(-10 + i)(-5 + 7i)$
98. $(-7 + 8i)(-1 + 7i)$
99. $(7 + i)(-10 + 10i)$
100. $(5 + 8i)(-1 - 6i)$
101. $(-4 + 3i)(-9 + 7i)$
102. $(7 + 10i)(-8 + 2i)$

103. $\dfrac{3+i}{7}$

104. $\dfrac{7-4i}{-4}$

105. $\dfrac{-4-6i}{-4i}$

106. $\dfrac{1-2i}{6i}$

107. $\dfrac{1}{1-8i}$

108. $\dfrac{1}{-2-5i}$

109. $\dfrac{-9i}{2+4i}$

110. $\dfrac{7i}{6+9i}$

111. $\dfrac{7+10i}{-4+8i}$

112. $\dfrac{-8+2i}{5-7i}$

113. $\dfrac{3-10i}{7+7i}$

114. $\dfrac{3+8i}{8-2i}$

115. $\dfrac{5-i}{-2+2i}$

116. $\dfrac{3+i}{7+i}$

Solve the following where z is a complex number and a and b are real:

117. $z - 8i = 10 - 3i$

118. $z + 2 = -2 - 6i$

119. $z + (7 + 3i) = -3i$

120. $z + (-10 + 9i) = -7 + 6i$

121. $z - 7 + 5i = -2 + 5i$

122. $z - i = 5i$

123. $z - 10 - i = -8 - 7i$

124. $z - 2 + 8i = 1 - 3i$

125. $a + 3i = -2 + 2bi$

126. $-3a + 8i = -9 + 5bi$

127. $10a + 9i = -5 - 2bi$

128. $-a - i = -6 + 2bi$

129. $a(-6 + 2bi) = -3 + 5i$

130. $a(-4b + 8i) = -7 - 3i$

131. $a(-4b - 4i) = 1 + 5i$

132. $a(-6 - 9bi) = -9 + 2i$

133. $\dfrac{z}{i} = 6 - 10i$

134. $\dfrac{z}{4i} = -5 + 10i$

135. $\dfrac{z}{-4 - 7i} = -6 - i$

136. $\dfrac{z}{-8 - 3i} = 2 + 6i$

137. $\dfrac{z}{-4 - 7i} = 3 - 6i$

138. $\dfrac{z}{-9 + 3i} = 1 - 2i$

139. $z(-4i) = -3 + 2i$

140. $z(3i) = -9 + 2i$

141. $z(-6 - 8i) = 7 + 2i$

142. $z(-5 - 9i) = -9i$

143. $z(-9 + 5i) = 10 - 7i$

144. $z(-9 - 2i) = 4 + 4i$

145. $\dfrac{5i}{z} = 8$

146. $\dfrac{-i}{z} = 10 + 4i$

147. $\dfrac{7-5i}{z} = 8-9i$

148. $\dfrac{5-3i}{z} = 2+i$

149. $\dfrac{-2+8i}{z} = 1$

150. $\dfrac{-6-8i}{z} = -5-9i$

For each of the following give a multiplier that results in a real product. (a and b are real numbers.)

Multiple answers may be correct for these questions.

151. $5-3i$

152. $-9+i$

153. $10a-7i$

154. $9+10ai$

155. $6a-bi$

156. $-3a-10bi$

157. Given $z\bar{z} = 25$, $z-\bar{z} = 8i$ and $\mathrm{Re}(z) > 0$, determine z.

158. Given $z\bar{z} = 169$, $z+\bar{z} = 10$ and $\mathrm{Im}(z) < 0$, determine z.

159. Solve $z+3\bar{z} = 12+2i$.

160. Solve $2z-\bar{z} = -3-3i$.

161. Solve $5z - 7\bar{z} = 14 + 24i$.

162. Solve $-2z - 6\bar{z} = 24 + 32i$.

163. Solve $9z - z\bar{z} = 17 + 9i$

164. Solve $z\bar{z} + 3z = 25 + 15i$

165. Solve $3z\bar{z} + z - \bar{z} = 75 - 6i$.

166. Solve $5z\bar{z} + 2(z - \bar{z}) = 70 + 10i$.

167. Solve $4z\bar{z} + 2z + 1 = 120 + 3i$.

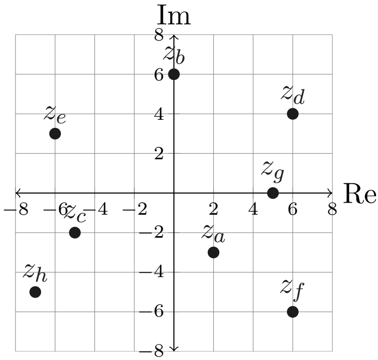
168. Solve $z\bar{z} - 6\bar{z} = 73 - 6i$.

169. Using $w = a + bi$ and $z = c + di$, prove $\overline{wz} = \bar{w}\bar{z}$

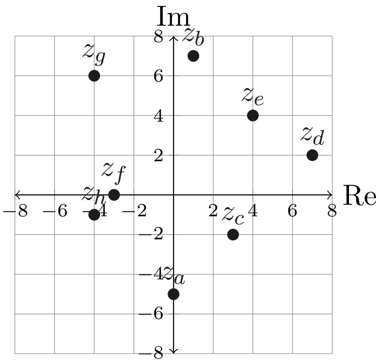
170. Using $w = a + bi$ and $z = c + di$, prove $\overline{\left(\frac{w}{z}\right)} = \frac{\bar{w}}{\bar{z}}$

1.5 The Complex Plane

171. Write the value of the points shown on the complex plane:



172. Write the value of the points shown on the complex plane:



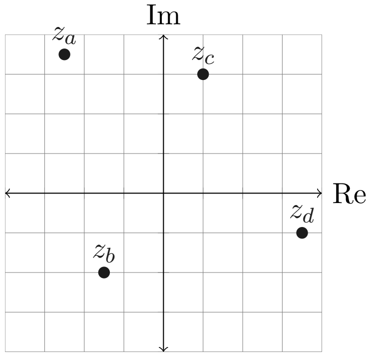
173. Plot these values on the complex plane:

$$z_a = -1 + 3i$$
$$z_b = 3 + 6i$$
$$z_c = -4 - i$$
$$z_d = -8 - 3i$$
$$z_e = 8 - 3i$$
$$z_f = 2 - 2i$$
$$z_g = -1 - 3i$$
$$z_h = 5i$$

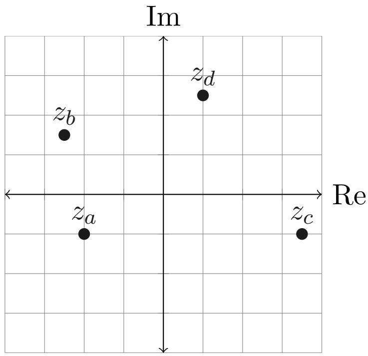
174. Plot these values on the complex plane:

$$z_a = -8 + 3i$$
$$z_b = 1$$
$$z_c = 7 - 3i$$
$$z_d = -5 + i$$
$$z_e = 5 + 2i$$
$$z_f = -4$$
$$z_g = 0$$
$$z_h = 7 + i$$

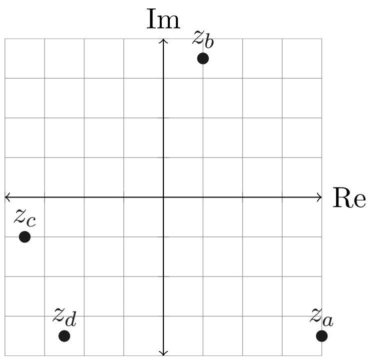
175. Plot the complex conjugate of the points shown:



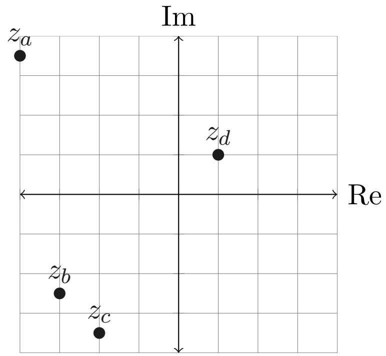
176. Plot the complex conjugate of the points shown:



177. Plot the complex conjugate of the points shown:



178. Plot the complex conjugate of the points shown:



179. Show how the sum $(4 + 2i) + (-5 + 4i)$ can be represented as a vector addition on the complex plane.

180. Show how the sum $(-3 + 4i) + (-1 - i)$ can be represented as a vector addition on the complex plane.

181. Given $z = -2-8i$, show how $z+\bar{z}$ can be represented as a vector addition on the complex plane.

182. Given $z = 3+5i$, show how $z + \bar{z}$ can be represented as a vector addition on the complex plane.

183. Given $z = -6-2i$, show how $z-\bar{z}$ can be represented as a vector addition on the complex plane.

184. Given $z = 7+3i$, show how $z - \bar{z}$ can be represented as a vector addition on the complex plane.

Plot the regions on the complex plane specified:

185. $\{z : \mathrm{Re}(z) > 2\}$

186. $\{z : \mathrm{Im}(z) < -4\}$

187. $\{z : \mathrm{Im}(z) \geqslant -5\}$

188. $\{z : \mathrm{Re}(z) \leqslant 4\}$

189. $\{z : -\frac{3\pi}{4} \leqslant \mathrm{Arg}(z) < -\frac{\pi}{4}\}$

190. $\{z : 0 \leqslant \mathrm{Arg}(z) \leqslant \frac{\pi}{6}\}$

191. $\{z : \frac{3\pi}{4} < \mathrm{Arg}(z) < \frac{5\pi}{4}\}$

192. $\{z : -\frac{\pi}{3} \leqslant \mathrm{Arg}(z) \leqslant \frac{4\pi}{3}\}$

193. $\{z : |z| = 7\}$

194. $\{z : |z| = 2\pi\}$

195. $\{z : |z| < 6\}$

196. $\{z : |z| \geqslant 2\}$

197. $\{z : 3 \leqslant |z| \leqslant 8\}$

198. $\{z : 6 \leqslant |z| < 7\}$

199. $\{z : |z - 2| = 5\}$

200. $\{z : |z + 3i| = 2\}$

201. $\{z : |z - 5i| \leqslant 4\}$

202. $\{z : |z + 4| > 5\}$

203. $\{z : 1 < |z - 3 + 4i| < 5\}$

204. $\{z : 2 \leqslant |z - 2.5 - 6i| < 3\}$

205. $\{z : 2\sqrt{2} < |z + 2 + 2i| < 8\}$

206. $\{z : 4\sqrt{2} < |z - 9 + 3i| \leqslant 8\}$

207. $\{z : |z| = |z - 7|\}$

208. $\{z : |z| = |z - 9i|\}$

209. $\{z : |z + 2i| = |z - 7|\}$

210. $\{z : |z + 5| = |z - 9i|\}$

211. $\{z : |z + 1| = |z - 5i|\}$

212. $\{z : |z - 3i| = |z + 4|\}$

213. $\{z : |z + 2i| = |z + 2 - 2i|\}$

214. $\{z : |z + 8 - 9i| = |z + 1 - 3i|\}$

215. $\{z : |z + 3 + 6i| \leqslant |z - 7 + 2i|\}$

216. $\{z : |z + 8 - 8i| \geqslant |z + 3 - 3i|\}$

217. $\{z : |z - 4 - 7i| > |z + 2 + 2i|\}$

218. $\{z : |z + 4 - 7i| < |z + 7 + 6i|\}$

219. $\{z : (|z + 2 + i| \leqslant 8) \cap (|z - 2 - 3i| \leqslant |z + 7 + 9i|)\}$

220. $\{z : (|z + 2 - 3i| < 6) \cap (|z - 9 - 5i| < |z + 6 + 2i|)\}$

221. $\{z : (|z + 3 - 3i| > 4) \cap (|z + 6| \leqslant |z - 7 - 3i|)\}$

222. $\{z : (|z + 5 + 2i| \geqslant 1) \cap (|z + 8 + 8i| < |z + 1 - 3i|)\}$

223. $\{z : (-\frac{2\pi}{3} < \text{Arg}(z) < -\frac{\pi}{3}) \cap (|z-3-2i| < |z-8+8i|)\}$

224. $\{z : (-\frac{\pi}{6} \leqslant \text{Arg}(z) \leqslant \pi) \cap (|z+6-8i| \leqslant |z+9+7i|)\}$

225. $\{z : (4 \leqslant |z-4-i| \leqslant 5) \cap (|z-3-4i| > |z-6-7i|)\}$

226. $\{z : (3 \leqslant |z+1| < 6) \cap (|z+3+4i| < |z-1+7i|)\}$

227. $\{z : (|z-4-2i| \leqslant 1) \cup (|z-4-2i| \geqslant 5)\}$

228. $\{z : (|z-2+i| \leqslant 3) \cup (|z-2+i| > 8)\}$

229. $\{z : (|z-5-3i| \leqslant 4) \cap (0 \leqslant \text{Arg}(z) \leqslant \frac{\pi}{4})\}$

230. $\{z : (|z-3| \leqslant 5) \cap (\frac{\pi}{12} \leqslant \text{Arg}(z) \leqslant \frac{\pi}{6})\}$

231. $\{z : (|z-5i| > 3) \cap (-\frac{\pi}{8} < \text{Arg}(z) < \frac{3\pi}{8})\}$

232. $\{z : (|z+4+i| > 8) \cap (-\frac{7\pi}{24} < \text{Arg}(z) < \frac{7\pi}{24})\}$

233. $\{z : |z-1-i| \leqslant |z-9+4i|) \cap (-\frac{3\pi}{8} < \text{Arg}(z) < \frac{5\pi}{24})\} \cap (2 < |z-4-4i| \leqslant 6)$

234. $\{z : |z+1-8i| \leqslant |z+4-2i|) \cap (0 < \text{Arg}(z) < \frac{\pi}{6})\} \cap (|z-5-5i| \leqslant 7)$

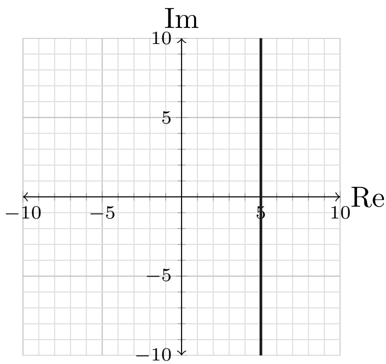
235. What is the area of the region of the Argand plane defined by $\{z : (|z-2-2i| < 5) \cap (|z-2-2i| > |z|)\}$?

236. What is the area of the region of the Argand plane defined by $\{z : (|z+2-5i| \leqslant 4) \cap (\frac{11\pi}{12} \leqslant \text{Arg}(z-2-5i) \leqslant \pi)\}$?

Give an equation or inequality, or a set of equations or inequalities, to define these regions of the complex plane:

Multiple answers may be correct for these questions.

237.



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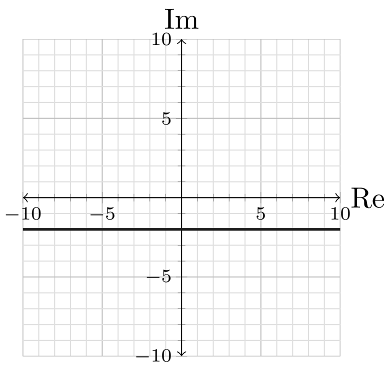

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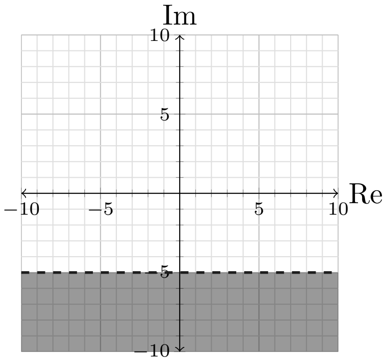

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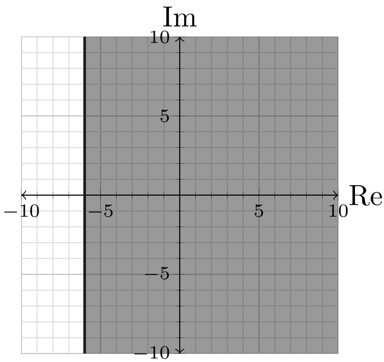

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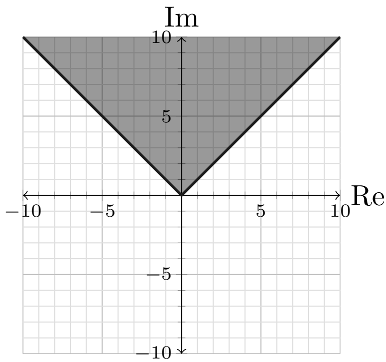

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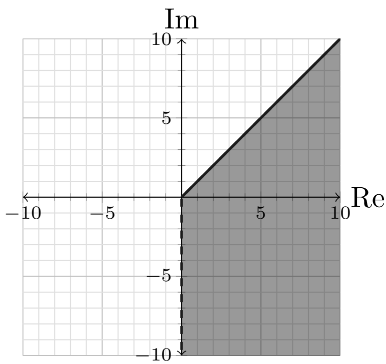

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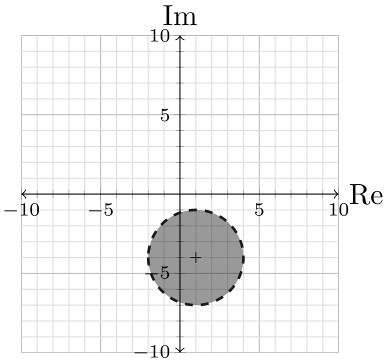

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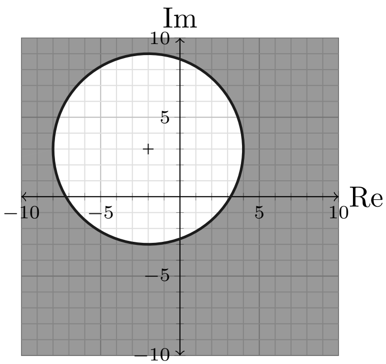

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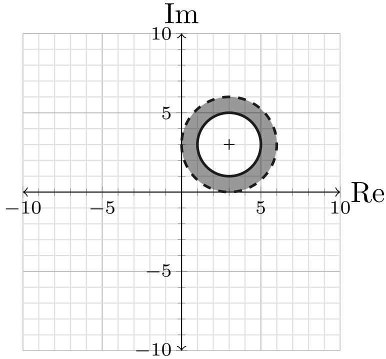

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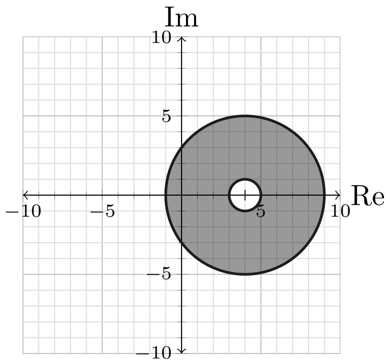

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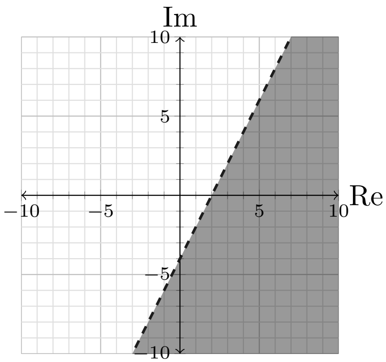

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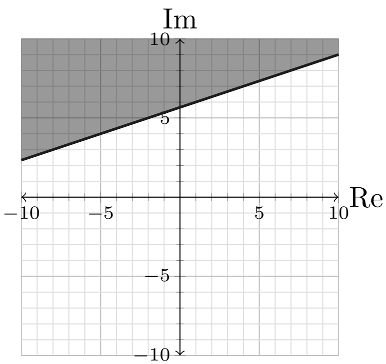

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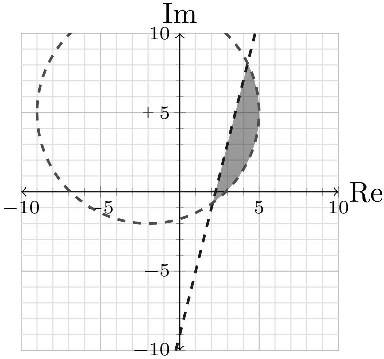

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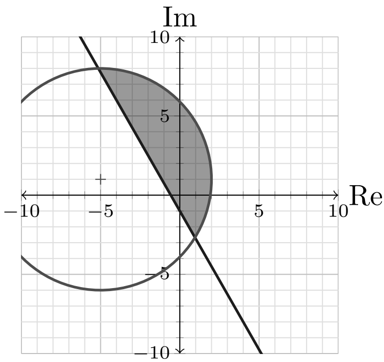

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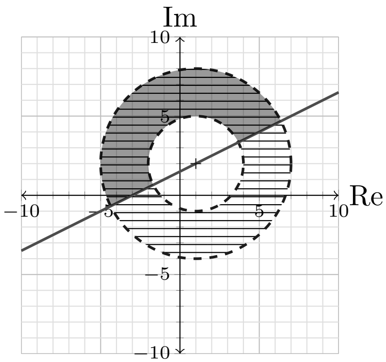

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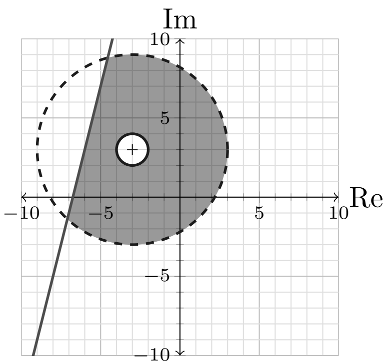

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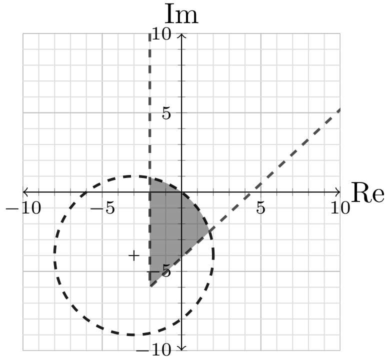

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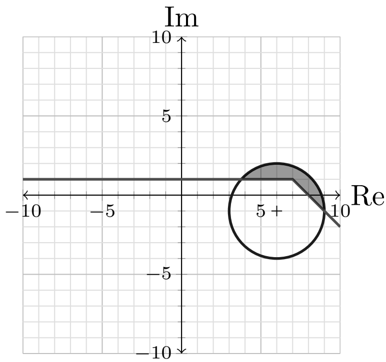

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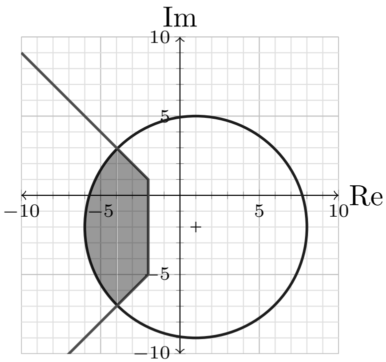
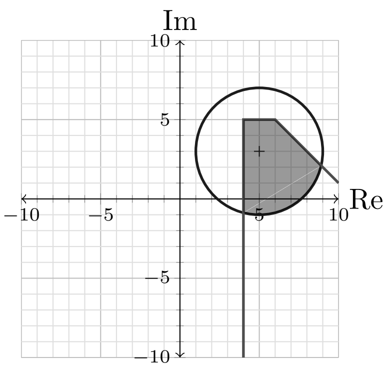


256.



1.6 Roots of Quadratic Equations

For each of the following quadratic functions, determine whether the roots are real or complex:

257. $f(x) = x^2 + 4x - 7$
258. $f(x) = x^2 + x + 8$
259. $f(x) = -x^2 - 9x - 6$
260. $f(x) = -x^2 + 2x + 3$
261. $f(x) = -4x^2 + 2x - 8$
262. $f(x) = 3x^2 + 5x - 8$
263. $f(x) = -2x^2 - 3x + 1$
264. $f(x) = -3x^2 - 8x - 6$

Determine the roots of the following quadratic functions:

265. $f(x) = x^2 + x + 1$
266. $f(x) = x^2 + 4x + 1$
267. $f(x) = -x^2 + x + 4$
268. $f(x) = -x^2 - 4$
269. $f(x) = 4x^2 - 5$
270. $f(x) = -2x^2 - 4x + 1$
271. $f(x) = -4x^2 + 10x + 10$
272. $f(x) = 3x^2 - 8x - 6$

Write the following quadratic expressions as the product of two linear factors (and, where appropriate, a constant factor):

273. $x^2 + 2x + 4$

274. $x^2 - 4x + 13$

275. $-x^2 + x + 3$

276. $-x^2 + 2x + 4$

277. $-2x^2 - 4x - 4$

278. $4x^2 + 20x + 29$

279. $4x^2 - x + 1$

280. $2x^2 - 7x + 7$

1.7 Factor and Remainder Theorems

For the following, decide if the linear expression q is a factor of the polynomial p. If it is not a factor, give the remainder of $\frac{p}{q}$.

281. $q = x - 3$, $p = x^3 - 8x^2 + 16x - 3$

282. $q = x - 7$, $p = x^3 - 9x^2 + 15x - 2$

283. $q = x + 8$, $p = x^3 - 50x + 112$

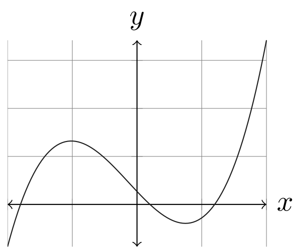
284. $q = x + 20$, $p = x^3 + 18x^2 - 39x + 40$

285. $q = x + 23$, $p = x^4 + 21x^3 - 46x^2 + x + 23$

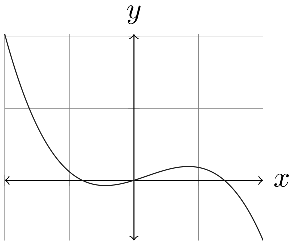
286. $q = x - 1$, $p = x^4 - 4x^3 + 6x^2 - 3x$
287. $q = x$, $p = x^4 + 2x^2 - 5x + 1$
288. $q = x + 13$, $p = x^4 - 168x^2 - 170$
289. $q = 3x - 9$, $p = 2x^3 - 27x - 27$
290. $q = 2x - 1$, $p = 2x^3 + 3x^2 - 4$
291. $q = 2x + 2$, $p = 3x^3 + 3x^2 + x + 1$
292. $q = 3x + 6$, $p = x^3 + 18x^2 - 39x + 20$
293. $q = 2x + 14$, $p = x^4 + 7x^3 - 2x^2 - 14x$
294. $q = 5x - 5$, $p = x^4 - 3x^3 + 2x^2 + 5$
295. $q = 3x$, $p = 3x^4 + x^3 - 5x - 9$
296. $q = 2x + 1$, $p = 2x^4 - 3x^3 - 3x - 2$

For the following graphs of polynomial functions, give the number of real roots and the number of pairs of complex conjugate roots.

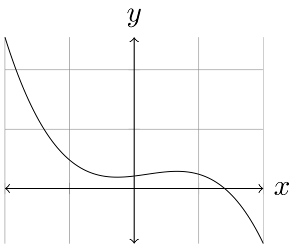
297. $f(x)$ is a cubic:



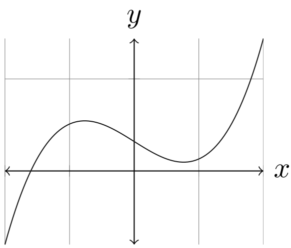
298. $f(x)$ is a cubic:



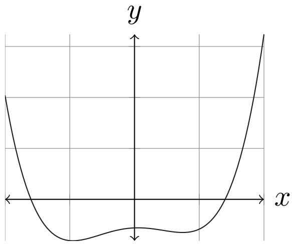
299. $f(x)$ is a cubic:



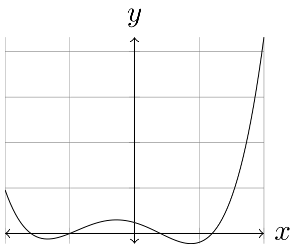
300. $f(x)$ is a cubic:



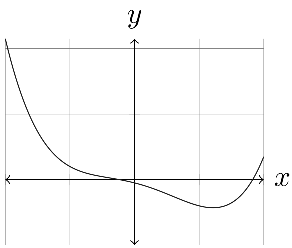
301. $f(x)$ is a quartic (i.e. 4th order):



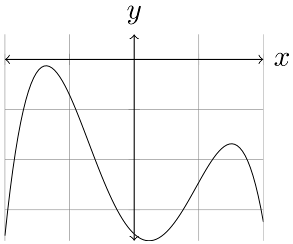
302. $f(x)$ is a quartic (i.e. 4th order):



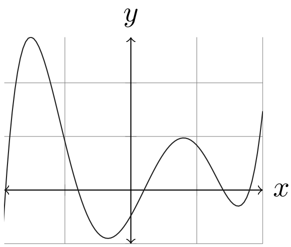
303. $f(x)$ is a quartic (i.e. 4th order):



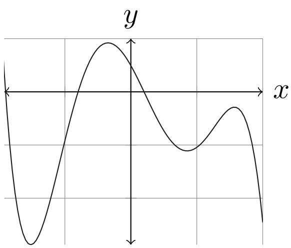
304. $f(x)$ is a quartic (i.e. 4th order):



305. $f(x)$ is a quintic (i.e. 5th order):



306. $f(x)$ is a quintic (i.e. 5th order):



307. Find the roots of $x^3 + x - 2$ given $x - 1$ is a factor.

308. Find the roots of $x^3 + 7x^2 + 15x + 25$ given $x + 5$ is a factor.

309. Find the roots of $x^3 - 5x^2 - 53x - 143$ given $x - 11$ is a factor.

310. Find the roots of $x^3 + 2x^2 + 10x - 36$ given $x - 2$ is a factor.

311. One of the roots of $x^3 + 6x^2 + 364x + 4040$ is $x = -10$. Find the other roots.

312. One of the roots of $x^3 - 10x^2 + 42x - 208$ is $x = 8$. Find the other roots.

313. One of the roots of $x^3 - 15x^2 + 67x - 117$ is $x = 3 + 2i$. Find the other roots.

314. One of the roots of $x^3 + 3x^2 + 9x + 27$ is $x = 3i$. Find the other roots.

315. One of the roots of $x^4 - 10x^3 - 50x^2 + 830x - 2331$ is $x = 6 - i$. Find the other roots.

316. One of the roots of $x^4 - 16x^3 + 124x^2 - 480x - 1600$ is $x = 4 + 8i$. Find the other roots.

317. One of the solutions of $4x^4 + 16x^3 + 29x^2 + 16x = -25$ is $x = -i$. Find the other solutions.

318. One of the solutions of $x^4 - 10x^3 + 25x^2 + 42x = -180$ is $x = 6 - 3i$. Find the other solutions.

319. One of the roots of $x^5 - 144x^3 + 8100x$ is $x = 9 + 3i$. Find the other roots.

320. One of the roots of $x^5 + 2x^4 - 57x^3 + 254x^2 + 3050x$ is $x = 6 - 5i$. Find the other roots.

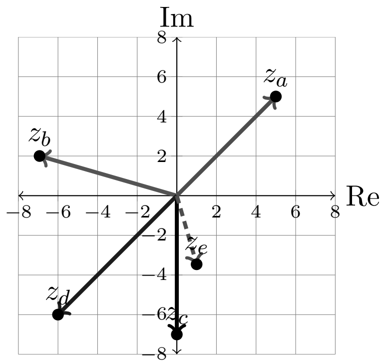
321. Two of the roots of $x^6 + 14x^5 + 85x^4 + 152x^3 - 220x^2 + 200x + 500$ are $x = -5 - 5i$ and $x = 1 - i$. Find the other roots.

322. Two of the roots of $x^6 - 2x^5 + x^4 - 8x^3 + 23x^2 + 10x - 25$ are $x = 2 + i$ and $x = -1 - 2i$. Find the other roots.

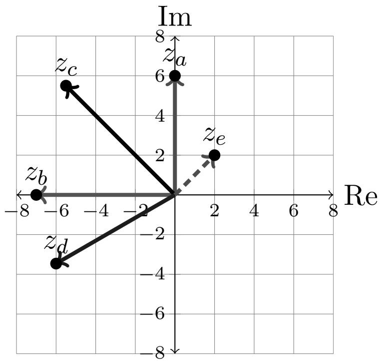
2 Complex Numbers in Polar Form

2.1 Argument

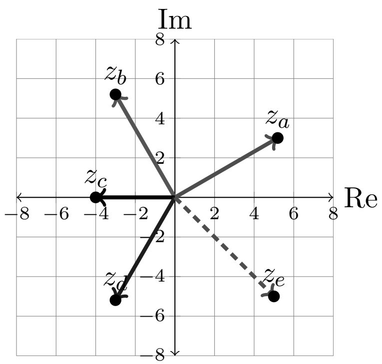
1. Estimate the argument (in degrees) of the complex numbers shown on the complex plane:



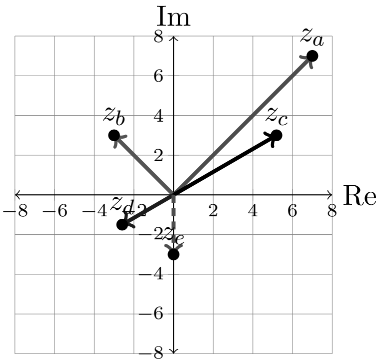
2. Estimate the argument (in degrees) of the complex numbers shown on the complex plane:



3. Estimate the argument (as fractions of π radians) of the complex numbers shown on the complex plane:



4. Estimate the argument (as fractions of π radians) of the complex numbers shown on the complex plane:



5. Sketch on the complex plane complex numbers having modulus 6 and arguments of:

a) 60°

b) 120°

c) 180°

d) 240°

e) 300°

f) 360°

6. Sketch on the complex plane complex numbers having modulus 6 and arguments of:

a) 30°

b) -30°

c) -90°

d) -150°

e) -210°

f) -270°

7. Sketch on the complex plane complex numbers having modulus 6 and arguments of:

a) $\frac{\pi}{6}$

b) $\frac{\pi}{4}$

c) $\frac{\pi}{3}$

d) $\frac{\pi}{2}$

e) $\frac{3\pi}{4}$

f) $\frac{5\pi}{6}$

8. Sketch on the complex plane complex numbers having modulus 6 and arguments of:

a) $-\frac{\pi}{4}$
b) $-\frac{5\pi}{6}$
c) $\frac{2\pi}{3}$
d) $-\frac{2\pi}{3}$
e) $-\frac{5\pi}{4}$
f) $-\frac{3\pi}{4}$

Determine the principal argument of a complex number specified with the following argument:

9. 390°
10. 700°
11. -920°
12. -1020°
13. 915°
14. 613°
15. $10\,915^\circ$
16. $51\,010^\circ$

17. 5π

18. -11π

19. $\frac{15\pi}{4}$

20. $\frac{25\pi}{3}$

21. $-\frac{19\pi}{6}$

22. $-\frac{21\pi}{4}$

23. $\frac{51\pi}{8}$

24. $-\frac{79\pi}{8}$

2.2 Converting Cartesian and Polar

For this section, answer with exact values where possible (without using a calculator), and round to three significant figures elsewhere.
Write the following in Cartesian form (i.e. $a + bi$):

25. $(\cos(30^\circ) + i\sin(30^\circ))$

26. $(\cos(45^\circ) + i\sin(45^\circ))$

27. $3\,(\cos(-60^\circ) + i\sin(-60^\circ))$

28. $5\,(\cos(135^\circ) + i\sin(135^\circ))$

29. $7\,(\cos(-100^\circ) + i\sin(-100^\circ))$

30. $20\,(\cos(155^\circ) + i\sin(155^\circ))$

31. $9\left(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})\right)$

32. $8\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$

33. $\left(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})\right)$

34. $\left(\cos(-\frac{5\pi}{6}) + i\sin(-\frac{5\pi}{6})\right)$

35. $9.25\,(\cos(0.655) + i\sin(0.655))$

36. $5.10\,(\cos(-2.27) + i\sin(-2.27))$

37. $\operatorname{cis}(150^\circ)$

38. $\operatorname{cis}(-120^\circ)$

39. $\sqrt{2}\operatorname{cis}(-45^\circ)$

40. $5\sqrt{2}\operatorname{cis}(225^\circ)$

41. $10\operatorname{cis}(87^\circ)$

42. $23.5\operatorname{cis}(-105^\circ)$

43. $8\operatorname{cis}(\pi)$

44. $10\operatorname{cis}(\frac{3\pi}{2})$

45. $\operatorname{cis}(-\frac{2\pi}{3})$

46. $\operatorname{cis}(\frac{7\pi}{3})$

47. $0.650\operatorname{cis}(0.09)$

48. $320\operatorname{cis}(1.54)$

Write the following in polar form with the angle specified in degrees:

49. $5 + 5i$

50. $8 + 8\sqrt{3}i$

51. $-7.5 + 7.5\sqrt{3}i$

52. $-3 - 3\sqrt{2}i$

53. $2 - 6i$

54. $-7 - 10i$

55. $1 - 9i$

56. $-6 + 10i$

Write the following in polar form with the angle specified in radians:

57. $5\sqrt{2} + 5\sqrt{2}i$

58. $\sqrt{3} + i$

59. $3\sqrt{3} - 9i$

60. $-9 + 9i$

61. $4 - 3i$

62. $-5 - i$

63. $8 + 2i$

64. $-7 - 5i$

2.3 Multiplying and Dividing in Polar Form

For each of the following find the product zw and the quotient $\frac{z}{w}$:

65. $z = 10\,\mathrm{cis}(20^\circ)$; $w = 2\,\mathrm{cis}(120^\circ)$

66. $z = 13\,\mathrm{cis}(45^\circ)$; $w = 10\,\mathrm{cis}(-75^\circ)$

67. $z = 14\,\mathrm{cis}(1.1)$; $w = 2\,\mathrm{cis}(1.2)$

68. $z = 12\,\mathrm{cis}(11)$; $w = 3\,\mathrm{cis}(-5)$

69. $z = 15\,\mathrm{cis}(0.22\pi)$; $w = 5\,\mathrm{cis}(-0.1\pi)$

70. $z = 20\,\mathrm{cis}(-1.1\pi)$; $w = 15\,\mathrm{cis}(0.9\pi)$

Solve:

71. $(8\,\mathrm{cis}(20^\circ))z = 64\,\mathrm{cis}(91^\circ)$

72. $(9\,\mathrm{cis}(55^\circ))z = 189\,\mathrm{cis}(11^\circ)$

73. $(12\,\mathrm{cis}(122^\circ))z = 240\,\mathrm{cis}(101^\circ)$

74. $(\mathrm{cis}(3.5^\circ))z = 21\,\mathrm{cis}(1.1^\circ)$

75. $(4\,\mathrm{cis}\,\frac{\pi}{6})z = 2\,\mathrm{cis}\,\frac{\pi}{4}$

76. $(18\,\mathrm{cis}\,\frac{5\pi}{6})z = 72\,\mathrm{cis}\,\frac{\pi}{4}$

77. $(7 \operatorname{cis} \frac{\pi}{2})z = 154 \operatorname{cis} \frac{5\pi}{6}$

78. $(6 \operatorname{cis} \frac{5\pi}{6})z = 18 \operatorname{cis}(-\frac{\pi}{4})$

79. $\frac{z(9 \operatorname{cis}(52^\circ))}{5 \operatorname{cis}(-17^\circ)} = 36$

80. $\frac{z(20 \operatorname{cis}(21^\circ))}{12 \operatorname{cis}(129^\circ)} = 60$

81. $\frac{z(3 \operatorname{cis}(\frac{5\pi}{6}))}{2 \operatorname{cis}(-\frac{\pi}{4})} = 51$

82. $\frac{z(5 \operatorname{cis}(-\frac{\pi}{2}))}{17 \operatorname{cis}(\frac{3\pi}{4})} = 85$

83. $\frac{z(7 \operatorname{cis}(2.99))}{2 \operatorname{cis}(-0.83)} = 21$

84. $\frac{z(25 \operatorname{cis}(1.22))}{7 \operatorname{cis}(0.54)} = 100$

For the following, solve for a and b where $a, b \in \mathbb{R}, a \geqslant 0, -\pi < b \leqslant \pi$ (or $-180 < b \leqslant 180$ for the questions with degrees).

85. $(a \operatorname{cis}(40^\circ))(11 \operatorname{cis}(b^\circ)) = -121i$

86. $(a \operatorname{cis}(-19^\circ))(6 \operatorname{cis}(b^\circ)) = 114i$

87. $(a \operatorname{cis} \frac{5\pi}{12})(14 \operatorname{cis}(b)) = 252i$

88. $(a \operatorname{cis}(-\frac{\pi}{6}))(23 \operatorname{cis}(b)) = -161i$

89. $(a \operatorname{cis}(21^\circ))(23 \operatorname{cis}(b^\circ)) = -529$

90. $(a \operatorname{cis}(23^\circ))(\operatorname{cis}(b^\circ)) = 14$

91. $(a \operatorname{cis}(-\frac{7\pi}{12}))(11 \operatorname{cis}(b)) = 187$

92. $(a\,\mathrm{cis}(\pi)(13\,\mathrm{cis}(b)) = -208$

2.4 Conjugates in Polar Form

Write the complex conjugate of z:

93. $z = 7\,\mathrm{cis}(177^\circ)$

94. $z = 3\,\mathrm{cis}(4^\circ)$

95. $z = 10\,\mathrm{cis}(0)$

96. $z = \mathrm{cis}(63^\circ)$

97. $z = 4\,\mathrm{cis}(\frac{-13\pi}{24})$

98. $z = 8\,\mathrm{cis}(\frac{13\pi}{24})$

99. $z = 10\,\mathrm{cis}(\frac{5\pi}{6})$

100. $z = 5\,\mathrm{cis}(\pi)$

Given $z = a + bi = r\,\mathrm{cis}\,\theta$, express w in terms of z and $\bar{z}$ without using $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\mathrm{Arg}(z)$ or $|z|$ (or their equivalent for $\bar{z}$, etc.):

101. $w = 3a - 3bi$

102. $w = 5a + 5bi$

103. $w = 4a$

104. $w = 10bi$

105. $w = 16r^2$

106. $w = 5\,\mathrm{cis}(2\theta)$

107. $w = 2a + 4bi$

108. $w = 5a - 3bi$

109. $w = -11a + 15bi$

110. $w = 9a - 20bi$

111. $w = 2a + 4b$

112. $w = (3a - 8b)i$

113. $w = r\,\mathrm{cis}(3\theta)$

114. $w = r^3\,\mathrm{cis}(5\theta)$

115. $w = r\,\mathrm{cis}(2\theta)$

116. $w = r^3\,\mathrm{cis}(-4\theta)$

2.5 De Moivre's Theorem

117. Use De Moivre's Theorem to show

$$\cos(3\theta) = \cos^3\theta - 3\sin^2\theta\cos\theta$$

118. Use De Moivre's Theorem to show

$$\sin(4\theta) = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$$

Simplify, leaving your answer in polar form:

119. $(4\,\text{cis}(15^\circ))^3$
120. $(3\,\text{cis}(28^\circ))^4$
121. $(2\,\text{cis}(-\frac{\pi}{6}))^5$
122. $(\sqrt{2}\,\text{cis}(\frac{\pi}{18}))^6$
123. $(\text{cis}(\frac{5\pi}{6}))^7$
124. $(\text{cis}(\frac{\pi}{4}))^9$
125. $(\text{cis}(\frac{\pi}{3}))^5$
126. $(\text{cis}(\frac{3\pi}{4}))^4$
127. $(5\,\text{cis}(\frac{\pi}{6}))^{-2}$
128. $(\sqrt{7}\,\text{cis}(-\frac{\pi}{6}))^{-4}$
129. $(2\,\text{cis}(\frac{5\pi}{6}))^{-3}$
130. $(3\,\text{cis}(-\frac{2\pi}{3}))^{-4}$

2.6 Complex Roots

On polar graph paper plot the solutions to $z^n = 1$ for the specified value of n.

131. $z^2 = 1$
132. $z^5 = 1$
133. $z^3 = 1$

134. $z^6 = 1$

135. $z^8 = 1$

136. $z^4 = 1$

List all the values that solve the following, giving answers in polar form:

137. $z^2 = 1$

138. $z^5 = 1$

139. $z^3 = 1$

140. $z^9 = 1$

141. $z^6 = 1$

142. $z^8 = 1$

143. $z^4 = 1$

144. $z^{10} = 1$

145. $z^7 = 1$

146. $z^{12} = 1$

147. $z^3 = i$

148. $z^5 = i$

149. $z^4 = i$

150. $z^6 = i$

151. $z^2 = -i$

152. $z^8 = -i$

153. $z^7 = -i$

154. $z^9 = -i$

155. $z^2 = \operatorname{cis} \frac{2\pi}{3}$

156. $z^3 = \operatorname{cis} \frac{\pi}{4}$

157. $z^5 = \operatorname{cis}(-\frac{5\pi}{6})$

158. $z^6 = \operatorname{cis} \frac{2\pi}{3}$

159. $z^3 = 8 \operatorname{cis} \frac{2\pi}{3}$

160. $z^5 = 9\sqrt{3} \operatorname{cis} \frac{\pi}{2}$

161. $z^9 = 512 \operatorname{cis}(135^\circ)$

162. $z^6 = 8 \operatorname{cis}(80^\circ)$

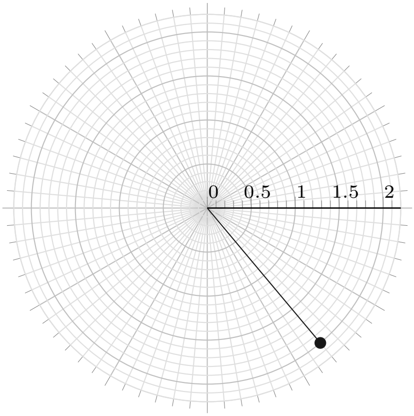
163. $z^2 = 49 \operatorname{cis}(-\frac{3\pi}{4})$

164. $z^6 = 64 \operatorname{cis}(-\frac{5\pi}{6})$

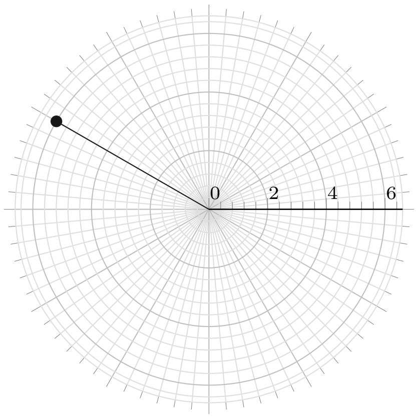
165. $z^8 = 81 \operatorname{cis} \frac{\pi}{4}$

166. $z^7 = 128 \operatorname{cis} \frac{7\pi}{8}$

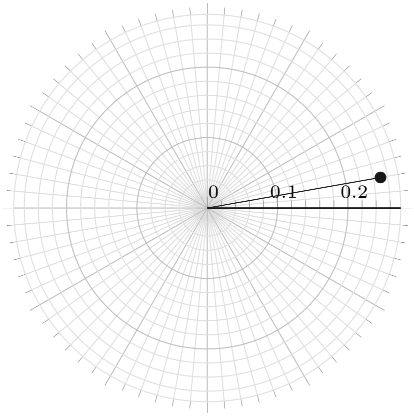
167. The graph below shows one solution to $z^4 = w$ for some complex w. Plot the other solutions.



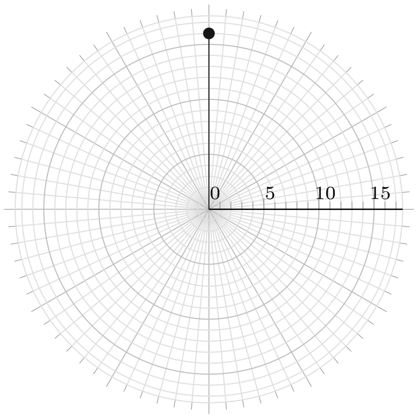
168. The graph below shows one solution to $z^3 = w$ for some complex w. Plot the other solutions.



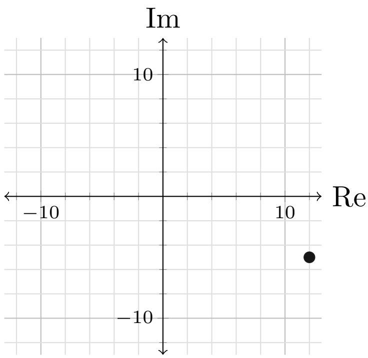
169. The graph below shows one solution to $z^8 = w$ for some complex w. Plot the other solutions.



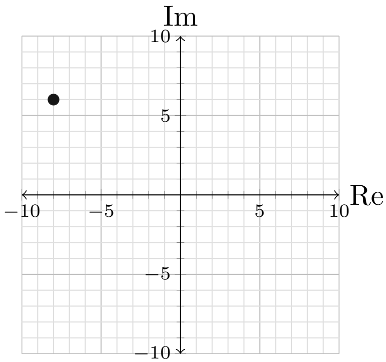
170. The graph below shows one solution to $z^6 = w$ for some complex w. Plot the other solutions.



171. The graph below shows one solution to $z^4 = w$ for some complex w. Plot the other solutions.



172. The graph below shows one solution to $z^2 = w$ for some complex w. Plot the other solutions.



2.7 EULER'S FORMULA

173. Show how Euler's formula can be used with index laws to demonstrate $(\operatorname{cis}\theta)^n = \operatorname{cis}(n\theta)$

174. Show how Euler's formula can be used with index laws to demonstrate $\frac{\operatorname{cis}\theta}{\operatorname{cis}\alpha} = \operatorname{cis}(\theta - \alpha)$

175. Show how Euler's formula can be used to write $\sin(x)$ in terms of e^{ix}.

176. Show how Euler's formula can be used to write $\cos(x)$ in terms of e^{ix}.

177. Expand $\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^5$ to obtain an expression for $\sin^5 x$ in terms of $\sin x$, $\sin 3x$ and $\sin 5x$.

178. Expand $\left(\frac{e^{ix}+e^{-ix}}{2}\right)^4$ to obtain an expression for $\cos^4 x$ in terms of $\cos 4x$ and $\cos 2x$.

Solutions

1. Complex Numbers as $a + bi$

1. real

3. real

5. imaginary

7. real

9. imaginary

11. real

13. complex

15. complex

17. imaginary

19. real

21. imaginary

23. imaginary

25. real

27. imaginary

29. real

31. real

33. $\mathbb{R}: x \geqslant 1$; $\mathbb{I}: x < 1$; $\mathbb{C}:$ none

35. $\mathbb{R}: x \leqslant \frac{9}{4}$; $\mathbb{I}:$ none; $\mathbb{C}: x > \frac{9}{4}$

37. $\mathbb{R}: 0 < x \leqslant 1$; $\mathbb{I}: x < 0$ or $x > 1$; $\mathbb{C}:$ none

39. $\mathbb{R}: x \geqslant 0$; $\mathbb{I}:$ none; $\mathbb{C}: x < 0$

41. $\mathbb{R}: x \leqslant -6$ or $x \geqslant 1$; $\mathbb{I}: -6 < x < 1$; $\mathbb{C}:$ none

43. $\mathbb{R}: x \geqslant 5$; $\mathbb{I}: x = 2$; $\mathbb{C}: x < 5, x \neq 2$

45. $\mathbb{R}: |x| \geqslant 4, x \neq -5$; $\mathbb{I}:$ none; $\mathbb{C}: -4 < x < 4$

47. $\mathbb{R}: x \geqslant 0$; $\mathbb{I}: x < -1$; $\mathbb{C}: -1 < x < 0$

49. $\sqrt{37}$

51. $2\sqrt{29}$

53. $5\sqrt{1 + a^2}$

55. $|a|\sqrt{25 + 16a^2}$

57. $\bar{z} = 3 - 4i$

59. $\bar{z} = -2 + \sqrt{3}i$

61. $\bar{z} = -29i$

63. $\bar{z} = -1.1i - 3$

65. $\bar{z} = \frac{5-\sqrt{25-4\times2\times6}}{2\times2} = \frac{5-\sqrt{23}i}{4}$

67. $\bar{z} = \frac{1}{r} - 5ci$

69. $8 - 5i$

71. $8 + 15i$

73. $-17.5 + 11.4i$

75. $10.2 - 1.3i$

77. $-14 - 5i$

79. $6 + 10i$

81. $13.8 - 1.6i$

83. $-0.8 - 12.5i$

85. $10i$

87. 24

89. $10 + 30i$

91. $-8 - 32i$

93. $-21 + 35i$

95. $13 - 115i$

97. $43 - 75i$

99. $-80 + 60i$

101. $15 - 55i$

103. $\frac{3}{7} + \frac{1}{7}i$

105. $\frac{3}{2} - i$

107. $\frac{1}{65} + \frac{8}{65}i$

109. $-\frac{9}{5} - \frac{9}{10}i$

111. $\frac{13}{20} - \frac{6}{5}i$

113. $-\frac{1}{2} - \frac{13}{14}i$

115. $-\frac{3}{2} - i$

117. $z = 10 + 5i$

119. $z = -7 - 6i$

121. $z = 5$

123. $z = 2 - 6i$

125. $a = -2,\ b = \frac{3}{2}$

127. $a=-\frac{1}{2}$, $b=-\frac{9}{2}$

129. $a=\frac{1}{2}$, $b=5$

131. $a=-\frac{5}{4}$, $b=\frac{1}{5}$

133. $z=10+6i$

135. $z=17+46i$

137. $z=-54+3i$

139. $z=-\frac{1}{2}-\frac{3}{4}i$

141. $z=-\frac{29}{50}+\frac{11}{25}i$

143. $z=-\frac{125}{106}+\frac{13}{106}i$

145. $z=\frac{5}{8}i$

147. $z=\frac{101}{145}+\frac{23}{145}i$

149. $z=-2+8i$

151. $5+3i$

153. $10a+7i$

155. $6a+bi$

157. $z=3+4i$

159. $z=3-i$

161. $z=-7+2i$

163. $z=6+i$ or $z=3+i$

165. $z=\pm 4-3i$

167. $z = 5 + \frac{3}{2}i$ or $z = -\frac{11}{2} + \frac{3}{2}i$

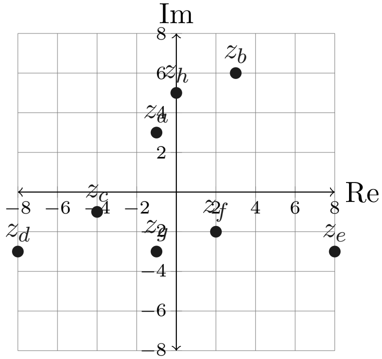
169. *Proof.*

$$\begin{aligned}
\text{L.H.S.} &= \overline{zw} \\
&= \overline{(a+bi)(c+di)} \\
&= \overline{ac + bdi^2 + (ad+bc)i} \\
&= ac + bdi^2 - (ad+bc)i \\
&= ac - adi - bci + bdi^2 \\
&= a(c-di) - bi(c-di) \\
&= (a-bi)(c-di) \\
&= \bar{w}\bar{z} = \text{R.H.S.}
\end{aligned}$$

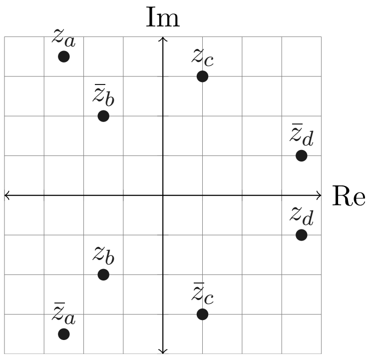
□

171. $z_a = 2 - 3i, z_b = 6i, z_c = -5 - 2i, z_d = 6 + 4i, z_e = -6 + 3i, z_f = 6 - 6i, z_g = 5, z_h = -7 - 5i$

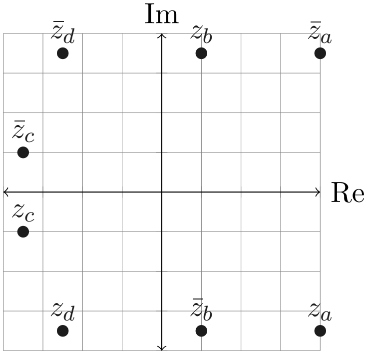
173.


Im
Re
8
6
4
2
−2
−4
−6
−8
−8
−6
−4
−2
2
4
6
8
z_a
z_b
z_c
z_d
z_e
z_f
z_g
z_h

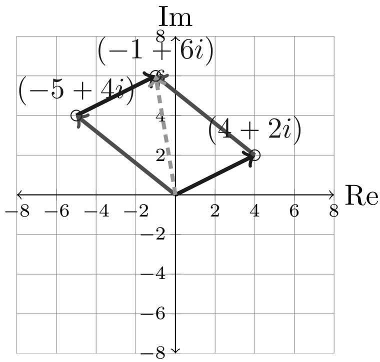
175.


Im
Re
z_a
$\bar{z}_b$
z_c
$\bar{z}_d$
z_d
z_b
$\bar{z}_c$
$\bar{z}_a$

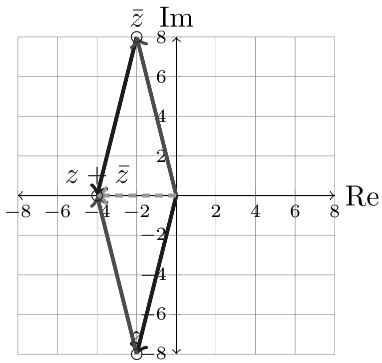
177.


Im
Re
$\bar{z}_d$
z_b
$\bar{z}_a$
$\bar{z}_c$
z_c
z_d
$\bar{z}_b$
z_a

179.

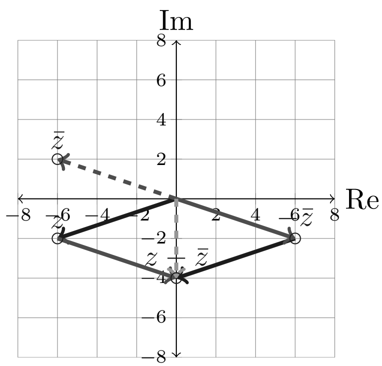

Im
Re
$(-1+6i)$
$(-5+4i)$
$(4+2i)$
−8 −6 −4 −2 2 4 6 8
8 4 2 −2 −4 −6 −8

181.



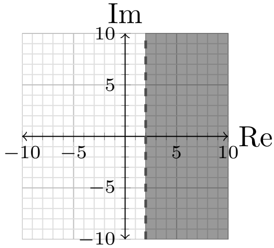
$z + \bar{z} = -4$

183.

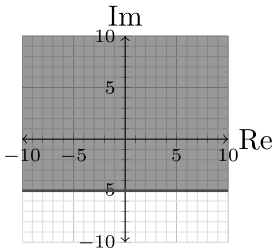


$z - \bar{z} = -4i$

185.



187.



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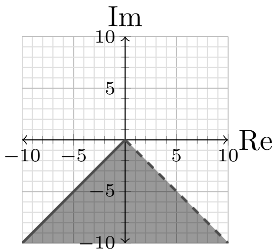

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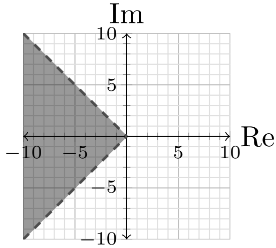

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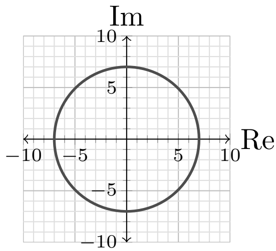

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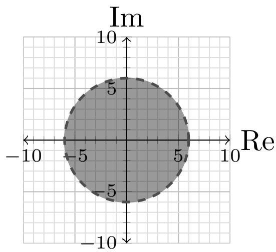

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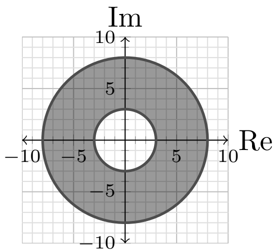

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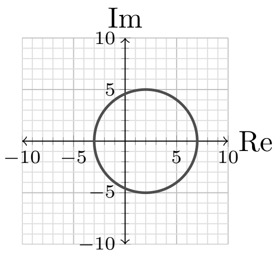

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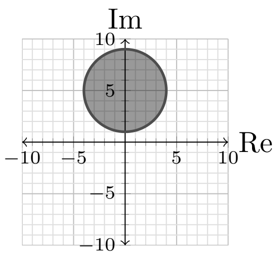

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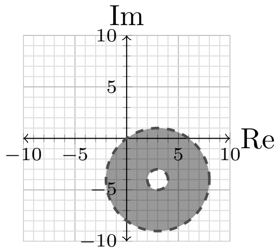

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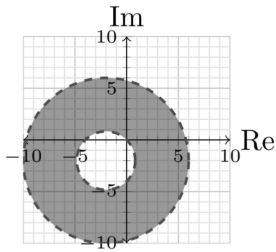

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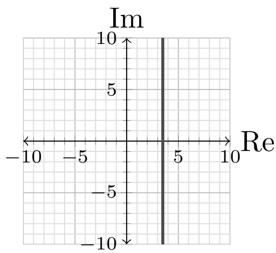

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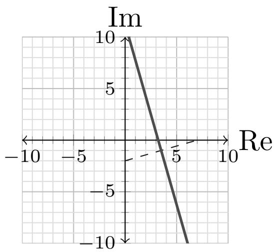
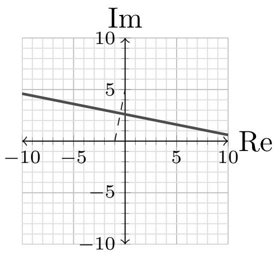


211.



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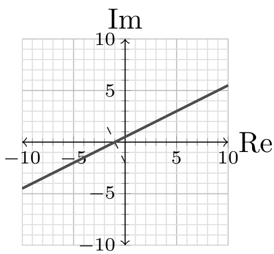

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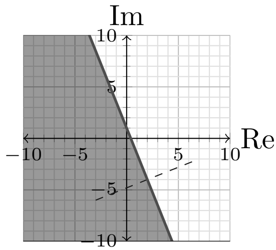

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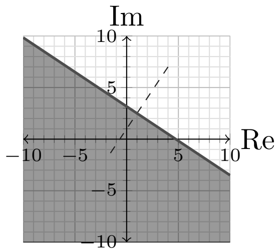

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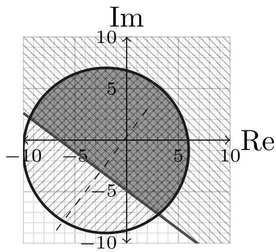

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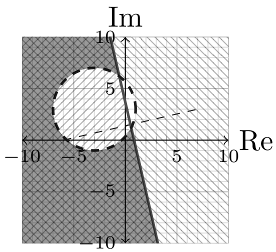

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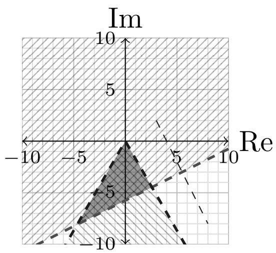

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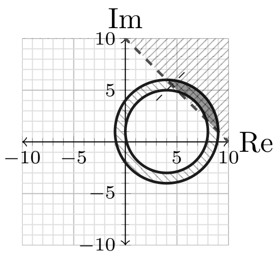

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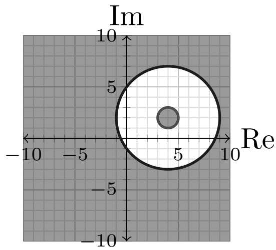

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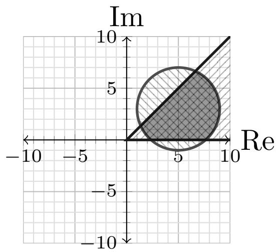

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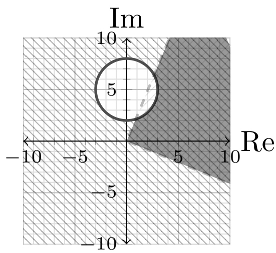

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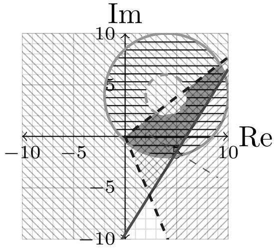

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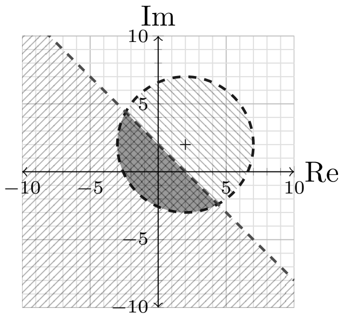


Distance from centre at $(2, 2)$ to line at $(1, 1)= \sqrt{2}$.
Angle subtended by the chord is given by

$$\cos\frac{\theta}{2} = \frac{\sqrt{2}}{5}$$

$$\theta = 2\cos^{-1}\frac{\sqrt{2}}{5}$$

and by Pythagoras

$$\begin{aligned}\sin\frac{\theta}{2} &= \sqrt{1-\frac{2}{25}}\\ &= \frac{\sqrt{23}}{5}\\ \therefore \sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ &= 2\times\frac{\sqrt{23}}{5}\times\frac{\sqrt{2}}{5}\\ &= \frac{2\sqrt{46}}{25}\end{aligned}$$

so the area of the segment is

$$\begin{aligned}A &= \frac{1}{2}r^2(\theta-\sin\theta)\\ &= \frac{25}{2}\Big(2\cos^{-1}\big(\frac{\sqrt{2}}{5}\big)-\frac{2\sqrt{46}}{25}\Big)\\ &\approx 25.32\text{units}^2\end{aligned}$$

237. $\{z : \mathrm{Re}(z) = 5\}$
239. $\{z : \mathrm{Im}(z) < -5\}$
241. $\{z : \frac{\pi}{4} \leqslant \mathrm{Arg}(z) \leqslant \frac{3\pi}{4}\}$

243. $\{z : |z - 1 + 4i| < 3\}$

245. $\{z : 2 \leqslant |z - 3 - 3i| < 3\}$

247. The solution should be of the form $\{z : |z - z_1| < |z - z_2|\}$ where z_2 is a reflection of z_1 in the line and z_1 is in the included region. Sample solution: $\{z : |z - (2 - 5i)| < |z - (-2 - 3i)|\}$

249. Sample solution: $\{z : (|z - (5 - 6i)| < |z - (-3 - 4i)|) \cap (|z - (-2 + 5i)| < 7)\}$

251. $\{z : (3 < |z - 1 - 2i| < 6) \cap (|z - 4i| < |z - 2|)\}$

253. $\{z : (|z + 3 + 4i| < 5) \cap (\frac{\pi}{4} < \mathrm{Arg}(|z + 2 + 6i|)\frac{\pi}{2}\}$

255. $\{z : (|z - 1 + 2i| \leqslant 7) \cap (\frac{3\pi}{4} \leqslant \mathrm{Arg}(|z - 1 + 2i|) \leqslant \frac{5\pi}{4}) \cap (\mathrm{Re}(z) \leqslant -2)\}$

257. real

259. real

261. complex

263. real

265. $x = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $x = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

267. $x = \frac{1}{2} - \frac{\sqrt{17}}{2}$, $x = \frac{1}{2} + \frac{\sqrt{17}}{2}$

269. $x = -\frac{\sqrt{5}}{2}$, $x = \frac{\sqrt{5}}{2}$

271. $x = \frac{5}{4} - \frac{\sqrt{65}}{4}$, $x = \frac{5}{4} + \frac{\sqrt{65}}{4}$

273. $(x+1-\sqrt{3}i)(x+1+\sqrt{3}i)$

275. $-(x-\frac{1}{2}-\frac{\sqrt{13}}{2})(x-\frac{1}{2}+\frac{\sqrt{13}}{2})$

277. $-2(x+1-i)(x+1+i)$

279. $4(x-\frac{1}{8}-\frac{\sqrt{15}}{8}i)(x-\frac{1}{8}+\frac{\sqrt{15}}{8}i)$

281. factor

283. factor

285. factor

287. remainder=1

289. remainder=−54

291. factor

293. factor

295. remainder=−9

297. Real roots: 3, pairs of complex conjugate roots:0

299. Real roots: 1, pairs of complex conjugate roots:1

301. Real roots: 2, pairs of complex conjugate roots:1

303. Real roots: 2, pairs of complex conjugate roots:1

305. Real roots: 5, pairs of complex conjugate roots:0

307. $x \in \left\{1, (-\frac{1}{2}-\frac{\sqrt{7}}{2}i), (-\frac{1}{2}+\frac{\sqrt{7}}{2}i)\right\}$

309. $x \in \{11, (-3-2i), (-3+2i)\}$

311. $x = 2 - 20i$ and $x = 2 + 20i$

313. $x = 3 - 2i$ and $x = 9$

315. $x = 6 + i$, $x = -9$ and $x = 7$

317. $x = i$, $x = -2 + \frac{3}{2}i$ and $x = -2 - \frac{3}{2}i$

319. $x = 9 - 3i$, $x = -9 - 3i$, $x = -9 + 3i$ and $x = 0$

321. $x = -5 + 5i$, $x = 1 + i$, $x = -5$ and $x = -1$

2. Complex Numbers in Polar Form

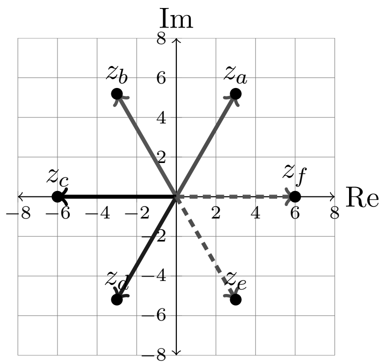
1.

$$\begin{aligned} \mathrm{Arg}(z_a) &= 45^\circ \\ \mathrm{Arg}(z_b) &= 150^\circ \\ \mathrm{Arg}(z_c) &= -90^\circ \\ \mathrm{Arg}(z_d) &= -135^\circ \\ \mathrm{Arg}(z_e) &= -60^\circ \end{aligned}$$

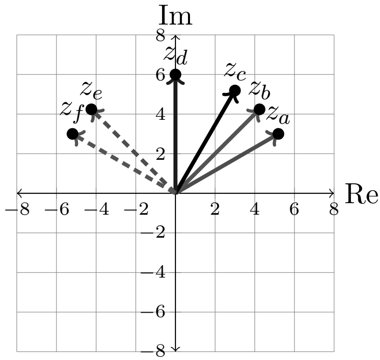
3.

$$
\begin{aligned}
\operatorname{Arg}(z_a) &= \frac{\pi}{6} \\
\operatorname{Arg}(z_b) &= \frac{2\pi}{3} \\
\operatorname{Arg}(z_c) &= \pi \\
\operatorname{Arg}(z_d) &= -\frac{2\pi}{3} \\
\operatorname{Arg}(z_e) &= -\frac{\pi}{4}
\end{aligned}
$$

5.



7.



9. 30°

11. 160°

13. -165°

15. 115°

17. π

19. $-\frac{\pi}{4}$

21. $\frac{5\pi}{6}$

23. $\frac{3\pi}{8}$

25. $\frac{\sqrt{3}}{2} + \frac{1}{2}i$

27. $\frac{3}{2} - \frac{3\sqrt{3}}{2}i$

29. $-1.22 - 6.89i$

31. $\frac{9}{2} + \frac{9\sqrt{3}}{2}i$

33. $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

35. $7.34 + 5.63i$

37. $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

39. $1 - i$

41. $0.523 + 9.99i$

43. -8

45. $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

47. $0.647 + 0.0584i$

49. $5\sqrt{2}\left(\cos(45^\circ) + i\sin(45^\circ)\right)$

51. $15\left(\cos(120^\circ) + i\sin(120^\circ)\right)$

53. $2\sqrt{10}\,\mathrm{cis}(-71.6^\circ)$

55. $\sqrt{82}\,\mathrm{cis}(-83.7^\circ)$

57. $10\left(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})\right)$

59. $6\sqrt{3}\left(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})\right)$

61. $5\,\mathrm{cis}(-0.644)$

63. $2\sqrt{17}\,\mathrm{cis}(0.245)$

65. $zw = 20\,\mathrm{cis}(140^\circ)$, $\frac{z}{w} = 5\,\mathrm{cis}(-100^\circ)$

67. $zw = 28\,\mathrm{cis}(2.3)$, $\frac{z}{w} = 7\,\mathrm{cis}(-0.1)$

69. $zw = 75\,\mathrm{cis}(0.12\pi)$, $\frac{z}{w} = 3\,\mathrm{cis}(0.32\pi)$

71. $z = 8\,\mathrm{cis}(71^\circ)$

73. $z = 20\,\mathrm{cis}(-21^\circ)$

75. $z = \frac{1}{2}\,\mathrm{cis}\,\frac{\pi}{12}$

77. $z = 22\,\mathrm{cis}\,\frac{\pi}{3}$

79. $z = 20\,\mathrm{cis}(-69^\circ)$

81. $z = 34\,\mathrm{cis}\,\frac{11\pi}{12}$

83. $z = 6\,\mathrm{cis}(2\pi - 3.82) \approx 6\,\mathrm{cis}(2.46)$

85. $a = 11$, $b = -130$

87. $a = 18$, $b = \frac{\pi}{12}$

89. $a = 23$, $b = 159$

91. $a = 17$, $b = \frac{7\pi}{12}$

93. $\bar{z} = 7\,\mathrm{cis}(-177^\circ)$

95. $\bar{z} = 10\,\mathrm{cis}(0)$

97. $\bar{z} = 4\,\mathrm{cis}(\frac{13\pi}{24})$

99. $\bar{z} = 10\,\mathrm{cis}(-\frac{5\pi}{6})$

101. $w = 3\bar{z}$

103. $w = 2(z + \bar{z})$

105. $w = 16z\bar{z}$

107. $w = 3z - \bar{z}$

109. $w = 2z - 13\bar{z}$

111. $w = (1-2i)z + (1+2i)\bar{z}$

113. $w = \frac{z^2}{\bar{z}}$

115. $w = \frac{z^{\frac{3}{2}}}{\bar{z}^{\frac{1}{2}}} = \sqrt{\frac{z^3}{\bar{z}}}$

117. *Proof.*

$$\begin{aligned}\operatorname{cis}(3\theta) &= \cos(3\theta) + i\sin(3\theta) \\ &= (\cos(\theta) + i\sin(\theta))^3 \\ &= \cos^3\theta + 3i\sin\theta\cos^2\theta - 3\sin^2\theta\cos\theta \\ &\quad - i\sin^3\theta\end{aligned}$$

Equating real components,

$$\cos(3\theta) = \cos^3\theta - 3\sin^2\theta\cos\theta$$

□

119. $64\operatorname{cis}(45^\circ)$

121. $32\operatorname{cis}(-\frac{5\pi}{6})$

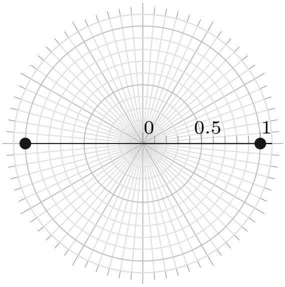
123. $\operatorname{cis}(-\frac{\pi}{6})$

125. $\operatorname{cis}(-\frac{\pi}{3})$

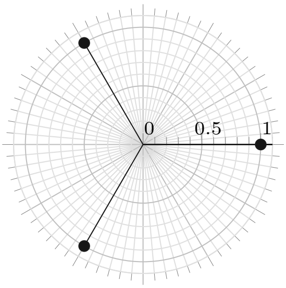
127. $0.04\operatorname{cis}(-\frac{\pi}{3})$

129. $\frac{1}{8}\operatorname{cis}(-\frac{\pi}{2})$

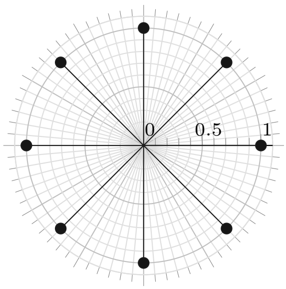
131.



133.



135.



137. $z = \operatorname{cis} 0, z = \operatorname{cis} \pi$

139. $z = \operatorname{cis} 0, z = \operatorname{cis} \frac{2\pi}{3}, z = \operatorname{cis} \frac{-2\pi}{3}$

141. $z = \operatorname{cis} 0, z = \operatorname{cis} \frac{\pi}{3}, z = \operatorname{cis} \frac{2\pi}{3}, z = \operatorname{cis}(-\frac{2\pi}{3}), z = \operatorname{cis}(-\frac{\pi}{3})$

143. $z = \operatorname{cis} 0, z = \operatorname{cis} \frac{\pi}{2}, z = \operatorname{cis} \pi, z = \operatorname{cis}(-\frac{\pi}{2})$

145. $z = \operatorname{cis} 0, z = \operatorname{cis} \frac{2\pi}{7}, z = \operatorname{cis} \frac{4\pi}{7}, z = \operatorname{cis} \frac{6\pi}{7}, z = \operatorname{cis}(-\frac{6\pi}{7}),$
$z = \operatorname{cis}(-\frac{4\pi}{7}), z = \operatorname{cis}(-\frac{2\pi}{7})$

147. $z \in \left\{\operatorname{cis} \frac{\pi}{6}, \operatorname{cis} \frac{5\pi}{6}, \operatorname{cis}(-\frac{\pi}{2})\right\}$

149. $z \in \left\{\operatorname{cis} \frac{\pi}{8}, \operatorname{cis} \frac{5\pi}{8}, \operatorname{cis}(-\frac{7\pi}{8}), \operatorname{cis}(-\frac{3\pi}{8})\right\}$

151. $z \in \left\{\operatorname{cis}(-\frac{\pi}{4}, \operatorname{cis} \frac{3\pi}{4}\right\}$

153. $z \in \left\{\operatorname{cis}(-\frac{\pi}{14}), \operatorname{cis} \frac{3\pi}{14}, \operatorname{cis} \frac{7\pi}{14}, \operatorname{cis} \frac{11\pi}{14}, \operatorname{cis}(-\frac{13\pi}{14}),\right.$
$\left.\operatorname{cis}(-\frac{9\pi}{14}), \operatorname{cis}(-\frac{5\pi}{14})\right\}$

155. $z \in \left\{\operatorname{cis}\frac{\pi}{3}, \operatorname{cis}(-\frac{2\pi}{3})\right\}$

157. $z \in \left\{\operatorname{cis}(-\frac{\pi}{6}),\ \operatorname{cis}\frac{7\pi}{30},\ \operatorname{cis}\frac{19\pi}{30},\ \operatorname{cis}(-\frac{29\pi}{30}),\ \operatorname{cis}(-\frac{17\pi}{30})\right\}$

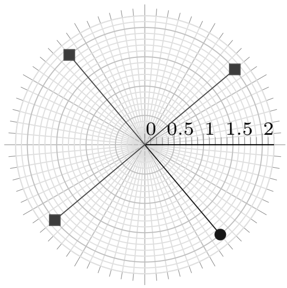
159. $z \in \left\{2\operatorname{cis}\frac{2\pi}{9}, 2\operatorname{cis}\frac{8\pi}{9}, 2\operatorname{cis}(-\frac{4\pi}{9})\right\}$

161. $z \in \left\{2\operatorname{cis}(15^\circ),\ 2\operatorname{cis}(55^\circ),\ 2\operatorname{cis}(95^\circ),\ 2\operatorname{cis}(135^\circ),\right.$
$\left.2\operatorname{cis}(175^\circ), 2\operatorname{cis}(-145^\circ), 2\operatorname{cis}(-105^\circ), 2\operatorname{cis}(-65^\circ), 2\operatorname{cis}(-25^\circ)\right\}$

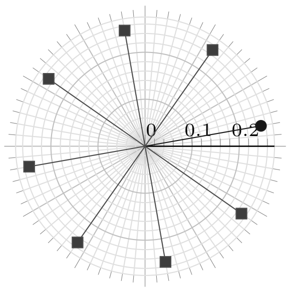
163. $z \in \left\{7\operatorname{cis}(-\frac{3\pi}{8}), 7\operatorname{cis}\frac{5\pi}{8}\right\}$

165. $z \in \left\{\sqrt{3}\operatorname{cis}\frac{\pi}{32},\ \sqrt{3}\operatorname{cis}\frac{9\pi}{32},\ \sqrt{3}\operatorname{cis}\frac{17\pi}{32},\ \sqrt{3}\operatorname{cis}\frac{25\pi}{32},\right.$
$\left.\sqrt{3}\operatorname{cis}(-\frac{31\pi}{32}),\ \sqrt{3}\operatorname{cis}(-\frac{23\pi}{32}),\ \sqrt{3}\operatorname{cis}(-\frac{15\pi}{32}),\ \sqrt{3}\operatorname{cis}(-\frac{7\pi}{32})\right\}$

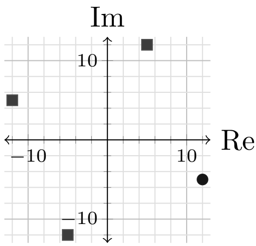
167.



169.



171.



173. *Proof.*

$$\begin{aligned}(\operatorname{cis}\theta)^n &= (e^{i\theta})^n \\ &= e^{in\theta} \\ &= \operatorname{cis}(n\theta)\end{aligned}$$

□

175.

$$\begin{aligned}\text{Let } z &= \cos x + i\sin x \\ \text{then } z - \bar{z} &= 2i\sin x \\ \therefore \quad \sin x &= \frac{z-\bar{z}}{2i}\end{aligned}$$

By Euler's formula, $z = e^{ix}$ and $\bar{z} = e^{-ix}$, so

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

177.

$$
\begin{aligned}
&\Big(\frac{e^{ix} - e^{-ix}}{2i}\Big)^5 \\
&\quad = \frac{1}{(2i)^5}\big(e^{5ix} - 5e^{4ix}e^{-ix} + 10e^{3ix}e^{-2ix} \\
&\qquad - 10e^{2ix}e^{-3ix} + 5e^{ix}e^{-4ix} - e^{5ix}\big) \\
&\quad = \frac{1}{2^5 i}\big(e^{5ix} - 5e^{3ix} + 10e^{ix} - 10e^{-ix} \\
&\qquad + 5e^{-3ix} - e^{5ix}\big) \\
&\quad = \frac{1}{2^5 i}\big((e^{5ix} - e^{5ix}) - 5(e^{3ix} - 5e^{-3ix}) \\
&\qquad + 10(e^{ix} - e^{-ix})\big) \\
&\quad = \frac{1}{2^4}\Big(\frac{e^{5ix} - e^{5ix}}{2i} - 5\frac{e^{3ix} - 5e^{-3ix}}{2i} \\
&\qquad + 10\frac{e^{ix} - e^{-ix}}{2i}\Big) \\
&\quad = \frac{1}{16}(\sin 5x - 5\sin 3x + 10\sin x)
\end{aligned}
$$